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# Lecture Notes for "Fourier Analysis and Differential Equations" [1] [2]

Time [3] & Place:

Tuesday, 13:30-15:05, 研教楼 306
(Sep. 8, Sep. 15, Sep. 20, Sep. 22, Sep. 29, Oct. 13, Oct. 20, Oct. 27, Nov. 3)

Wednesday, 10:05-11:40, 综合教学 1 号楼 356
(Sep. 9, Sep. 16, Sep. 23, Sep. 30, Oct. 10, Oct. 14, Oct. 21, Oct. 28, Nov. 4)

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[1] The lecture notes are based on a lecture given at the Dalian University of Technology during the fall term 2026.
[2] Comments are welcome to be sent to me by email.
[3] It takes place from Sep. 6 to Nov. 6, 2026 (2.-10. semester week).

# Contents

This two-month lecture provides an introduction to the sophisticated theory of Fourier analysis. The one-dimensional case is particularly emphasized, serving as motivation and providing examples for the illustration of the general theory. It also helps to connect the various concepts encountered in the theories of ordinary differential equations (ODEs), functional analysis, complex analysis, harmonic analysis and many other fields. Due to time constraints and the ambitious goal, we are unable to define all the related mathematical concepts rigorously, and instead aim to draw the audience's interest to the relevant theory. (The footnotes are left as exercises to the audience.) Finally the application to partial differential equations (PDEs) is demonstrated.

# 1 Fourier series

## 1.1 Examples of boundary value problems

### 1.1.1 An ODE with Dirichlet boundary condition

Consider the boundary value problem for a second-order ordinary differential equation on the interval $[0, \pi]$,

$$u''(x) + \lambda u(x) = 0 \text{ on } (0, \pi), \quad u(0) = u(\pi) = 0. \tag{1.1}$$

We consider three different cases:

- Case $\lambda < 0$.

  The following two functions are the fundamental solutions for the ODE $u''(x) = -\lambda u(x)$

  $$\sinh(\sqrt{-\lambda}x) = \frac{e^{\sqrt{-\lambda}x} - e^{-\sqrt{-\lambda}x}}{2},$$
  $$\cosh(\sqrt{-\lambda}x) = \frac{e^{\sqrt{-\lambda}x} + e^{-\sqrt{-\lambda}x}}{2},$$

  The boundary conditions imply that the solution should be zero. [4]

- Case $\lambda = 0$.

  A regular solution of $u''(x) = 0$ is a linear function $u(x) = ax + b$, which is 0 due to the boundary conditions.

[4] One can also argue from another perspective: We take the $L^2(0, \pi)$-inner product

- Case $\lambda > 0$.

  The following two functions are fundamental solutions for the ODE $u''(x) = -\lambda u(x)$

$$\sin(\sqrt{\lambda}x) = \frac{e^{i\sqrt{\lambda}x} - e^{-i\sqrt{\lambda}x}}{2i},$$
$$\cos(\sqrt{\lambda}x) = \frac{e^{i\sqrt{\lambda}x} + e^{-i\sqrt{\lambda}x}}{2},$$

  The boundary conditions imply that the solution should be a linear combination of

$$\sin(nx), \quad n = 1, 2, \cdots$$

To conclude, there is an infinite set of discrete values of $\lambda$:

$$\lambda = n^2, \quad n = 1, 2, \cdots$$

such that the boundary value problem (1.1) has a <u>nontrivial</u> solution. These values $n^2$, $n \in \mathbb{N}$ are called the eigenvalues of the (eigenvalue) problem (1.1) and the functions

$$u_n(x) = b_n \sin(nx), \quad n \in \mathbb{N}, \quad b_n \in \mathbb{R}(\text{or } \mathbb{C})$$

are the corresponding eigenfunctions. The concept of eigenvalues and eigenfunctions becomes clearer in Section 1.2 below.

### 1.1.2 Fourier's idea to solve the heat equation

Consider the one-dimensional initial-boundary value problem for the (real-valued) heat equation

$$\begin{cases} \partial_t u - \partial_{xx} u = 0 \text{ on } (0,\infty) \times (0,\pi), \\ u(0,x) = f(x) \quad x \in (0,\pi), \\ u(t,0) = u(t,\pi) = 0 \quad t \in (0,\infty), \end{cases} \tag{1.3}$$

between the equation and the solution $u$ itself, to obtain by integration by parts

$$-\int_0^\pi |u'|^2 \,\mathrm{d}x + \lambda \int_0^\pi |u|^2 \,\mathrm{d}x = 0. \tag{1.2}$$

This implies, if $u$ is a regular solution (e.g. in $H^1(0,\pi)$), then $u = 0$. This argument can be easily generalized to the case when $\lambda \in \mathbb{C}$ with $\operatorname{Im} \lambda \neq 0$, where $u = 0$ (in $H^1(0,\pi)$) follows.

where the compatibility condition is satisfied

$$f(0) = f(\pi) = 0,$$

and we assume that $f$ is not identically zero: $f \neq 0$.
J. Fourier[5] made the following Ansatz (separation of variables)

$$u(t, x) = T(t)X(x).$$

This leads to the following:

- $T'(t)X(x) - T(t)X''(x) = 0$, which implies (at least formally),

$$\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = -\lambda \text{ for some constant } \lambda.$$

- $T(t)X(0) = T(t)X(\pi) = 0$ for all time $t$, which implies

$$X(0) = X(\pi) = 0.$$

- $T(0)X(x) = f(x)$.

From the ODE results above, if $f \neq 0$ and one searches for nontrivial (real-valued) solution, $\lambda$ can "only" take the value $n^2$ for some $n \in \mathbb{N}$, and $X(x)$ is a multiple of $\sin(nx)$, and in this case, the time function $T(t) = T(0)e^{-n^2 t}$. That is, if the initial data is

$$f(x) = b_n \sin(nx),$$

for some $n \in \mathbb{N}$ and some constant $b_n$, then the problem (1.3) has a (unique) solution

$$u(t, x) = T(t)X(x) = T(0)e^{-n^2 t}\frac{b_n \sin(nx)}{T(0)} = b_n e^{-n^2 t} \sin(nx).$$

If the initial data is a finite sum of sine functions

$$f(x) = \sum_{n=1}^{N} b_n \sin(nx),$$

then by the superposition principle the solution is

$$u(t, x) = \sum_{n=1}^{N} b_n e^{-n^2 t} \sin(nx).$$

[5] Joseph Fourier, *Théorie Analytique de la Chaleur.* Chez F. Didot, 1822 (Firstly presented to *Institute de France* in 1807).

Thus, formally, if we can expand the initial data $f(x)$ as a sine series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx), \tag{1.4}$$

then the following function (when it is well-defined) is a solution of (1.1)

$$u(x) = \sum_{n=1}^{\infty} b_n e^{-n^2 t} \sin(nx).$$

A function with such an expansion (1.4) (nowadays called Fourier sine series) "seems" very special. For half a century around 1800 there was a great debate whether such a series expansion holds only for specific functions.

## 1.2 Spectral theorem and Fourier series

We claim that any function in the functional space

$$\{f \in C^2([0,\pi]) \mid f(0) = f(\pi) = 0\} \tag{1.5}$$

can be expanded as Fourier sine series. Historically one can make use of the Weierstrass approximation theorem and the density of trigonometric functions in the space of polynomial functions to prove this statement. Here we illustrate this in modern mathematical language.
We can reformulate (1.1) as the following eigenvalue problem

$$\mathcal{A}u = \lambda u, \tag{1.6}$$

where the operator $\mathcal{A}$ is defined by[6]

$$\mathcal{A} : D(\mathcal{A}) \to X, \quad \mathcal{A}(u) = -u'' \text{ for } u \in D(\mathcal{A}). \tag{1.7}$$

Here we take the underlying inner product space and scalar product below

$$X = C([0,\pi];\mathbb{C}), \quad \langle u, v\rangle = \int_0^{\pi} u\bar{v}\,\mathrm{d}x\,,$$

and the domain of the operator $\mathcal{A}$ is a subspace of $X$:

$$D(\mathcal{A}) = \{u \in C^2([0,\pi]) \mid u(0) = u(\pi) = 0\} \subset X.$$

[6]There is a certain conceptual mismatch between the classical setting of continuous functions and the spectral-theoretic framework, which is more naturally aligned with spaces such as $L^2(0,\pi)$. For example, one may define $\mathcal{A} : H_0^2(0,\pi) = D(\mathcal{A}) \to L^2(0,\pi)$.

The induced norm on $X$ is

$$\|u\| = \langle u, u\rangle^{\frac{1}{2}},$$

and the distance between $u$ and $v$ is $\|u - v\|$. Note that $D(\mathcal{A})$ and $X$ are <u>not</u> complete with respect to the defined metric.
Recall that the Lebesgue space

$$\begin{aligned} L^2(0,\pi) =& \{f \text{ is Lebesgue measurable on } (0,\pi) \mid \\ & \|f\|_{L^2(0,\pi)} = \Big(\int_0^\pi |f(x)|^2 \,\mathrm{d}x\Big)^{\frac{1}{2}} < \infty\}/\sim, \end{aligned}$$

where

- The integral is understood in the sense of Riemann-Lebesgue:

$$\|f\|_{L^2(0,\pi)} = \Big(2\int_0^\infty \alpha\ m(\{x \in (0,\pi) \mid |f(x)| > \alpha\})\, d\alpha\Big)^{\frac{1}{2}}$$

- The equivalence $\sim$ means identity almost everywhere (a.e.):

$$f \sim g \Leftrightarrow m(\{x \in (0,\pi) \mid f(x) \neq g(x)\}) = 0.$$

The Lebesgue space $L^2(0,\pi)$ is endowed with the scalar product $\langle\cdot,\cdot\rangle$, and is a Hilbert space. By the density of the space[7] of test functions $C_c^\infty(0,\pi)$ in $L^2(0,\pi)$, the closure of $C_c^\infty(0,\pi) \subset D(A) \subset X$ in the Lebesgue space $(L^2(0,\pi), \langle\cdot,\cdot\rangle)$ is the Lebesgue space itself

$$\overline{X}^{L^2} = \overline{D(\mathcal{A})}^{L^2} = \overline{C_c^\infty(0,\pi)}^{L^2} = L^2(0,\pi).$$

### 1.2.1 Resolvent computation

From the ODE results above (recalling (1.2)), the eigenvalues and normalized eigenfunctions of the operator $\mathcal{A}$ are

$$\lambda_n = n^2, \quad u_n = \sqrt{\frac{2}{\pi}}\sin(nx), \quad n \in \mathbb{N}, \tag{1.8}$$

with $\|u_n\|_{L^2(0,\pi)} = 1$.

[Sep 8, 2026]

[Sep 9, 2026]

[7] The density result also holds on the whole line or in higher dimensions.

We now compute the resolvent of $\mathcal{A}$ at $\lambda = -1 \notin \mathbb{N}$, that is, we compute $R_{\mathcal{A}}(-1) = (\mathcal{A}+1)^{-1} : X \to D(\mathcal{A})$. That is, for a given function $g \in X$, we seek the solution $u \in D(\mathcal{A})$ of the following boundary value problem for the inhomogeneous ODE:

$$-u'' + u = g, \quad u(0) = u(\pi) = 0. \tag{1.9}$$

It suffices to compute the Green function of this boundary value problem. The homogeneous equation $-u'' + u = 0$ has fundamental solutions

$$e^{-x}, \quad e^{x}.$$

The solution satisfying the boundary condition $u(0) = 0$ is a multiple of

$$\phi(x) = e^{x} - e^{-x},$$

while the solution satisfying the boundary condition $u(\pi) = 0$ is a multiple of

$$\psi(x) = e^{-(\pi-x)} - e^{\pi-x}.$$

Their Wronskian $W(\phi,\psi) = \phi\psi' - \phi'\psi = \det\begin{pmatrix}\phi & \psi \\ \phi' & \psi'\end{pmatrix}$ is independent of $x$ [8] and is given by

$$W(\phi,\psi) = 2(e^{\pi} - e^{-\pi}).$$

The Green function[9] is

$$\begin{aligned} G(x,y) &= \begin{cases} -\frac{\phi(x)\psi(y)}{W(\phi,\psi)} & \text{for } x < y \\ \\ -\frac{\phi(y)\psi(x)}{W(\phi,\psi)} & \text{for } x > y \end{cases} \\ &= -\frac{e^{-\pi+x+y} + e^{\pi-(x+y)} - e^{\pi-|x-y|} - e^{-\pi+|x-y|}}{W(\phi,\psi)}. \end{aligned}$$

Thus the unique solution of (1.9) reads[10]

$$u(x) = \int_0^{\pi} G(x,y) g(y) \, \mathrm{d}y\,, \tag{1.10}$$

[8] It follows from Abel's identity (also called Liouville's formular).

[9] The $-$ sign before the quantity $\frac{\phi(x)\psi(y)}{W(\phi,\psi)}$ or $\frac{\phi(y)\psi(x)}{W(\phi,\psi)}$ in the definition of the Green function here is due to the $-$ sign before the highest order derivative $\partial_{xx}$ in the equation (1.9), which (usually) is assumed to be $+$ in the standard formulation of ODE.

[10] The unique solvability of the inhomogeneous problem (1.9) follows from the fact that the homogeneous problem (1.1) with $\lambda = -1$ has only trivial zero solution, by the Fredholm

that is,

$$R_{\mathcal{A}}(-1)g(x) = (\mathcal{A}+1)^{-1}g(x) = \int_0^\pi G(x,y)g(y)\,\mathrm{d}y\,.$$

One may verify that

- $R_{\mathcal{A}}(-1) : X \to D(\mathcal{A})$.

  This follows immediately from (1.9).

- $R_{\mathcal{A}}(-1)$ is a compact operator in $X$. That is, for any bounded sequence $(g_k)_k$ in $X$, the function sequence $(R_{\mathcal{A}}(-1)g_k)_k$ has a subsequence which converges in $X$.

  This follows from the properties of $G$ and a compactness argument, e.g. Arzela-Ascoli theorem.

- $R_{\mathcal{A}}(-1)$ is symmetric. That is,

$$\big\langle R_{\mathcal{A}}(-1)g, h\big\rangle = \big\langle g, R_{\mathcal{A}}(-1)h\big\rangle, \quad \forall g, h \in X.$$

  This follows from the symmetry property of $G(x,y) = G(y,x) \in \mathbb{R}$.

### 1.2.2 Spectral theorem and Fourier sine series

Spectral theorem[11] states that given an infinte-dimensional inner product space $X$ and a compact symmetric operator $\mathcal{R} : X \to X$, there exists a sequence of real eigenvalues $(z_n)_n \subset \mathbb{R}$ converging to 0. The corresponding normalized eigenvectors $(v_n)_n$ form an orthonormal set and every $f \in \mathrm{Ran}(\mathcal{R})$ can be written as

$$f = \sum_{n=1}^{\infty} \langle f, v_n\rangle v_n.$$

If $\mathrm{Ran}(\mathcal{R})$ is dense, then the eigenvectors form an orthonormal basis.

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alternative theorem.

The unique solution of (1.9) can be represented by Green function (1.10): $u(x) = \int_0^x G(x,y)g(y)\,\mathrm{d}y + \int_x^\pi G(x,y)g(y)\,\mathrm{d}y$, where the Green function has the following properties

- $G(x,y) \in C([0,\pi]^2)$, $G(x,y) \in C^2(\Delta_\pm)$ with $\Delta_\pm = \{(x,y) \in [0,\pi]^2 \mid \pm(x-y) \leqslant 0\}$, and $\partial_x G$ has a jump at the diagonal line: $\partial_x G(x,x-0) - \partial_x G(x,x+0) = -1$;
- $G(\cdot, y)$ satisfies the homogeneous ODE (with respect to $x$ variable) in $\Delta_\pm$ and the boundary conditions.

[11] We don't give the proof here.

Thus for the compact symmetric operator $R_{\mathcal{A}}(-1)$ in $X$, there exists a sequence of real eigenvalues $(z_n) \to 0$ and the corresponding normalized eigenfunctions

$$(v_n) \subset D(\mathcal{A}), \quad \|v_n\| = \langle v_n, v_n \rangle = 1,$$

which form an orthonormal basis in the closure of its range $\overline{\mathrm{Ran}(R_{\mathcal{A}}(-1))}^{L^2} = \overline{D(\mathcal{A})}^{L^2} = L^2(0,\pi)$. That is, for any $f \in L^2(0,\pi)$, we can expand it as follows:

$$f = \sum_{n=1}^{\infty} \langle f, v_n \rangle v_n.$$

Now we return to the operator $\mathcal{A}$. The fact that $R_{\mathcal{A}}(-1)v_n = z_n v_n$ is equivalent to

$$\mathcal{A}v_n = (-1 + \frac{1}{z_n})v_n,$$

that is, $v_n$ is a normalized eigenfunction of $\mathcal{A}$ associated with the eigenvalue $-1 + \frac{1}{z_n}$ which tends to infinity. Recalling (1.8), we find

$$\lambda_n = n^2 = -1 + \frac{1}{z_n}, \quad u_n = \sqrt{\frac{2}{\pi}} \sin(nx) = v_n.$$

To conclude, the expansion $f = \sum_{n=1}^{\infty} \langle f, v_n \rangle v_n$ for any function $f \in L^2(0,\pi)$ is indeed (1.4):

$$f = \sum_{n=1}^{\infty} b_n \sin(nx), \quad b_n = \sqrt{\frac{2}{\pi}} \langle f, v_n \rangle = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx)\, \mathrm{d}x\,.$$

This is the Fourier sine series. Let us emphasize here that the series is converges with respect to our scalar product, i.e. in the sense of $L^2(0,\pi)$. We remark that if the function $f \in D(\mathcal{A})$, then the sine series converges uniformly to $f$. The convergence of Fourier series is discussed in more detail in Sections 1.3 and 1.4 later.

[Sep 9, 2026]

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[Sep 15, 2026]

### 1.2.3 Eigenvalue problem with Neumann boundary condition

Instead of Dirichlet boundary condition studied in Section 1.2, we consider the Laplace eigenvalue problem associated to the Neumann boundary condition. More precisely, we consider the eigenvalue problem

$$\mathcal{B}u = \lambda u, \quad \mathcal{B} : D(\mathcal{B}) \to X, \quad \mathcal{B}(u) = -u'' \text{ for } u \in D(\mathcal{B}), \tag{1.11}$$

where the domain of the operator $\mathcal{B}$ is

$$D(\mathcal{B}) = \{u \in C^2([0,\pi]) \,|\, u'(0) = u'(\pi) = 0\} \subset X,$$

with $X = C([0,\pi];\mathbb{C})$ equipped with the inner product $\langle \cdot, \cdot \rangle$.
The same analysis as in Subsection 1.1.1 yields the following eigenvalues and normalized eigenfunctions

$$\lambda_0 = 0, \quad \lambda_n = n^2, \quad u_0 = \sqrt{\frac{1}{\pi}}, \quad u_n = \sqrt{\frac{2}{\pi}}\cos(nx), \quad n \in \mathbb{N}\,.$$

We can then compute the resolvent $R_{\mathcal{B}}(-1) : X \to D(\mathcal{B})$ in terms of Green function

$$R_{\mathcal{B}}(-1)h(x) = \int_0^\pi G_B(x,y)h(y)\,\mathrm{d}y\,, \quad G_{\mathcal{B}}(x,y) = \begin{cases} -\frac{\phi_{\mathcal{B}}(x)\psi_{\mathcal{B}}(y)}{W(\phi_{\mathcal{B}},\psi_{\mathcal{B}})} & \text{for } x \leqslant y \\ \\ -\frac{\phi_{\mathcal{B}}(y)\psi_{\mathcal{B}}(x)}{W(\phi_{\mathcal{B}},\psi_{\mathcal{B}})} & \text{for } x \geqslant y \end{cases}$$

and verify that it is compact and symmetric, as in Subsection 1.2.1[12]. Finally the spectral theorem ensures that any function $f \in L^2(0,\pi)$ can be expanded as Fourier cosine series

$$\begin{aligned} &f = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx), \\ &a_0 = \frac{1}{\pi}\int_0^\pi f(x)\,\mathrm{d}x\,, \quad a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos(nx)\,\mathrm{d}x\,, \quad n \in \mathbb{N}, \end{aligned}$$

in the sense of $L^2(0,\pi)$.

### 1.2.4 Full Fourier series

We now consider the eigenvalue problem

$$\mathcal{C}u = \lambda u, \quad \mathcal{C} : D(\mathcal{C}) \to Y, \quad \mathcal{C}(u) = -u'' \text{ for } u \in D(\mathcal{C}), \tag{1.12}$$

where

- $Y = C([-\pi,\pi];\mathbb{C})$ is equipped with the inner product $\langle \cdot, \cdot \rangle_{L^2(-\pi,\pi)}$;
- we assume periodic boundary conditions on the interval $[-\pi,\pi]$

$$D(\mathcal{C}) = \{u \in C^2([-\pi,\pi]) \,|\, u(-\pi) = u(\pi), \quad u'(-\pi) = u'(\pi)\} \subset Y.$$

[12] We omit the details here.

There are eigenvalues

$$\lambda_0 = 0, \quad \lambda_n = n^2, \quad n \in \mathbb{N},$$

and the corresponding normalized eigenfunctions

$$u_0 = \sqrt{\frac{1}{2\pi}}, \quad \{u_n, v_n\} = \left\{\sqrt{\frac{1}{\pi}}\cos(nx),\ \sqrt{\frac{1}{\pi}}\sin(nx)\right\}, \quad n \in \mathbb{N}\,.$$

The spectral theorem implies that any function $f \in L^2(-\pi,\pi)$ can be expanded as Fourier series $\underline{\text{in the sense of } L^2(-\pi,\pi)}$:

$$f = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx), \quad a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x\,,$$
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,\mathrm{d}x\,, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,\mathrm{d}x\,, \quad n \in \mathbb{N}\,.$$

Observe that

- If $f \in C([-\pi,\pi])$ is an odd function, then $a_0 = a_n = 0$, and hence the full Fourier series of any odd function on $(-\pi,\pi)$ coincides with Fourier sine series on $(0,\pi)$ in the sense of $L^2(0,\pi)$.

  Conversely, if $f \in C([0,\pi])$ with Dirichlet boundary condition $f(0) = f(\pi) = 0$, then one can extend it to an odd function in $C([-\pi,\pi])$ satisfying $f(-\pi) = f(0) = f(\pi) = 0$, and the Fourier sine series coincides with the full Fourier series on $(0,\pi)$ in the sense of $L^2(0,\pi)$.

  This is the physical intuition why the Fourier sine series is "complete" on $[0,\pi]$.

- If $f \in C([-\pi,\pi])$ is an even function, then $b_n = 0$, and hence the full Fourier series of any even function on $(-\pi,\pi)$ coincides with Fourier cosine series on $(0,\pi)$ in the sense of $L^2(0,\pi)$.

  Conversely, if $f \in C([0,\pi])$ with Neumann boundary condition $f'(0) = f'(\pi) = 0$, then one can extend it to an even function in $C([-\pi,\pi])$ satisfying $f'(-\pi) = f'(0) = f'(\pi) = 0$, and the Fourier cosine series coincides with full Fourier series on $(0,\pi)$ in the sense of $L^2(0,\pi)$.

- Replacing

$$\cos(nx) = \frac{1}{2}(e^{inx} + e^{-inx}), \quad \sin(nx) = \frac{1}{2i}(e^{inx} - e^{-inx}),$$

we can rewrite the full Fourier series in the elegant complex form

$$
\begin{aligned}
&f = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} c_n e^{inx}, \quad \frac{1}{\sqrt{2\pi}} c_0 = a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-i0x} \, \mathrm{d}x \,, \\
&\frac{1}{\sqrt{2\pi}} c_n = \frac{1}{2} a_n + \frac{1}{2i} b_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} \, \mathrm{d}x \,, \quad n \in \mathbb{N}, \\
&\frac{1}{\sqrt{2\pi}} c_{-n} = \frac{1}{2} a_n - \frac{1}{2i} b_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-i(-n)x} \, \mathrm{d}x \,, \quad n \in \mathbb{N} \,.
\end{aligned}
$$

## 1.3 Pointwise convergence of Fourier series

The spectral theorem guarantees the convergence of Fourier series

$$
f(x) = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} c_n e^{inx}, \quad c_n = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(x) e^{-inx} \, \mathrm{d}x \tag{1.13}
$$

in the sense of $L^2(-\pi, \pi)$. There is a long history of mathematicians searching for the convergence of the partial sum of the Fourier series

$$
S_N f(x) = \frac{1}{\sqrt{2\pi}} \sum_{n=-N}^{N} c_n e^{inx}, \quad c_n = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(x) e^{-inx} \, \mathrm{d}x \tag{1.14}
$$

as $N \to \infty$, at first in the sense of pointwise convergence.

### 1.3.1 Dirichlet kernel

The first positive results is due to P.G.L. Dirichlet (1829). He rewrote the partial sum of a $2\pi$-periodic function as

$$
\begin{aligned}
S_N f(x) &= \frac{1}{2\pi} \sum_{n=-N}^{N} \int_{-\pi}^{\pi} f(t) e^{-int} \, \mathrm{d}t \, e^{inx} \\
&= \frac{1}{2\pi} \sum_{n=-N}^{N} \int_{-\pi}^{\pi} f(t) e^{in(x-t)} \, \mathrm{d}t \\
&= \int_{-\pi}^{\pi} f(t) D_N(x-t) \, \mathrm{d}t = \int_{-\pi}^{\pi} f(x-t) D_N(t) \, \mathrm{d}t = (f * D_N)(x),
\end{aligned}
$$

where $D_N$ is the so-called Dirichlet kernel

$$
D_N(y) = \frac{1}{2\pi} \sum_{n=-N}^{N} e^{iny}
$$

$$
\begin{aligned}
&= \frac{1}{2\pi}\big(1+2\cos y+2\cos(2y)+\cdots+2\cos(Ny)\big)\\
&= \frac{1}{2\pi}\frac{1}{\sin(\frac{1}{2}y)}\Big(\sin(\frac{1}{2}y)+(\sin(\frac{3}{2}y)-\sin(\frac{1}{2}y))+(\sin(\frac{5}{2}y)-\sin(\frac{3}{2}y))+\\
&\qquad +\cdots+\big(\sin((N+\frac{1}{2})y)-\sin((N-\frac{1}{2})y)\big)\Big)\\
&= \frac{\sin((N+\frac{1}{2})y)}{2\pi\sin(\frac{1}{2}y)}.
\end{aligned}
$$

Observe that $D_N(y)$ satisfies

$$
\int_{-\pi}^{\pi} D_N(y)\,\mathrm{d}y\; = 1 \text{ and } |D_N(y)| \leqslant \frac{1}{2\pi\sin(\frac{1}{2}\delta)} \text{ if } \delta \leqslant |y| \leqslant \pi. \tag{1.15}
$$

Notice that $\int_{-\pi}^{\pi}|D_N(y)|\,\mathrm{d}y$ is growing in terms of $N$ of the rate $lnN$[13]:

$$
\begin{aligned}
&\int_{-\pi}^{\pi}|D_N(y)|\,\mathrm{d}y\; = \frac{1}{2\pi}\int_{-\pi}^{\pi}\Big|\frac{\sin((N+\frac{1}{2})y)}{\sin(\frac{1}{2}y)}\Big|\,\mathrm{d}y\; = \frac{1}{\pi}\int_0^{\pi}\frac{|\sin((N+\frac{1}{2})y)|}{\frac{1}{2}y}\,\mathrm{d}y\; + O(1)\\
&= \frac{2}{\pi}\int_0^{(N+\frac{1}{2})\pi}\frac{|\sin t|}{t}\,\mathrm{d}t\; + O(1) = \frac{2}{\pi}\sum_{k=0}^{N-1}\int_{k\pi}^{(k+1)\pi}\frac{|\sin t|}{t}\,\mathrm{d}t\; + O(1)\\
&= \frac{2}{\pi}\sum_{k=0}^{N-1}\int_0^{\pi}\frac{|\sin t|}{t+k\pi}\,\mathrm{d}t\; + O(1)\\
&= \frac{2}{\pi}\int_0^{\pi}\sin t\Big(\sum_{k=0}^{N-1}\frac{1}{t+k\pi}\Big)\,\mathrm{d}t\; + O(1) = O(\ln N).
\end{aligned} \tag{1.16}
$$

We claim that the pointwise convergence of a Fourier series at some point $x$ depends on the regularity/smoothness of the function at $x$, and hence is (surprisingly) a <u>local</u> property (cf. Riemann's "second theorem" in his Habilitation thesis[14]), although the definition of Fourier series depends on the values of this function on the whole interval $[-\pi,\pi]$. To see this, consider first the special case $f \in L^2(-\pi,\pi)$ which satisfies $f=0$ on a neighborhood of $x$: $(x-\delta,x+\delta)$, and we compute

$$
S_N f(x) = \int_{\delta\leqslant|y|\leqslant\pi} f(x-y)\frac{\sin((N+\frac{1}{2})y)}{2\pi\sin(\frac{1}{2}y)}\,\mathrm{d}y
$$

[13]This growth is the reason for the failure of the pointwise convergence of Fourier series for continuous functions, see du Bois-Reymond's negative result later.

[14]Bernhard Riemann, Ueber die Darstellbarkeit einer Function durch eine trigonometrische Reihe. Abhandlungen der Königlichen Gesellschaft der Wissenschaften zu Göttingen 13, 87–132, 1867. (Habilitationsschrift, Göttingen, 1854)

$$
\begin{aligned}
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-y)\frac{1}{\sin(\frac12 y)} 1_{\{\delta\leqslant|y|\leqslant\pi\}}\big(\frac{1}{2i}(e^{(N+\frac12)iy}-e^{-(N+\frac12)iy})\big)\,\mathrm{d}y\\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\Big(f(x-y)\frac{1}{\sin(\frac12 y)}1_{\{\delta\leqslant|y|\leqslant\pi\}}\frac{1}{2i}e^{\frac12 iy}\Big)e^{iNy}\,\mathrm{d}y\\
&\quad -\frac{1}{2\pi}\int_{-\pi}^{\pi}\Big(f(x-y)\frac{1}{\sin(\frac12 y)}1_{\{\delta\leqslant|y|\leqslant\pi\}}\frac{1}{2i}e^{-\frac12 iy}\Big)e^{-iNy}\,\mathrm{d}y\,.
\end{aligned}
$$

This is the difference of the Fourier coefficients of two integrable functions at $-N$ and $N$, both of which tends to 0 as $N \to \infty$: At this point $x$, by the Riemann-Lebesgue lemma[15],

$$\lim_{N\to\infty} S_N f(x) = 0 = f(x).$$

### 1.3.2 Dini's criterion and du Bois-Reymond's counterexample

Now we consider the function $f$ with general behavior on $(x-\delta, x+\delta)$ and compute (due to the unit size of the integral of $D_N$)

$$S_N f(x) - f(x) = \int_{-\pi}^{\pi}\big(f(x-y)-f(x)\big)\frac{\sin((N+\frac12)y)}{2\pi\sin(\frac12 y)}\,\mathrm{d}y \;= \int_{|y|<\delta} + \int_{\delta\leqslant|y|\leqslant\pi}.$$

**Dini's criterion** If we assume the following continuity condition for the function $f$ at the point $x$:

$$\int_{|y|<\delta}\Big|\frac{f(x-y)-f(x)}{y}\Big|\,\mathrm{d}y \;< \infty, \tag{1.17}$$

for some $\delta > 0$, then all the functions

$$\frac{f(x-y)-f(x)}{\sin(\frac12 y)}1_{\{|y|<\delta\}}\frac{1}{2i}e^{\pm\frac12 iy}, \qquad \frac{f(x-y)-f(x)}{\sin(\frac12 y)}1_{\{\delta\leqslant|y|\leqslant\pi\}}\frac{1}{2i}e^{\pm\frac12 iy}$$

[15] For any integrable function $g$ which is periodic on $\mathbb{R}$ with period $2\pi$, we compute

$$
\begin{aligned}
\hat g(N) := \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} g(y)e^{-iNy}\,\mathrm{d}y \;&= -\frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} g(y)e^{-iN(y+\frac{\pi}{N})}\,\mathrm{d}y\\
= -\frac{1}{\sqrt{2\pi}}\int_{-\pi+\frac{\pi}{N}}^{\pi+\frac{\pi}{N}} g(t-\frac{\pi}{N})e^{-iNt}\,\mathrm{d}t \;&= -\frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} g(y-\frac{\pi}{N})e^{-iNy}\,\mathrm{d}y\,,
\end{aligned}
$$

and thus

$$\hat g(N) = \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} g(y)e^{-iNy}\,\mathrm{d}y \;= \frac12\frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi}\big(g(y)-g(y-\frac{\pi}{N})\big)e^{-iNy}\,\mathrm{d}y\,.$$

If $g$ is continuous, then its Fourier coefficient $\hat g(N)$ tends to 0 as $N$ tends to infinity. If $g \in L^1(-\pi,\pi)$, then we can approximate it with some continuous function to derive the convergence result.

are integrable on $[-\pi,\pi]$, and hence Riemann-Lebesgue lemma implies

$$S_N f(x) \to f(x) \text{ as } N \to \infty.$$

The continuity property (1.17) is called the Dini's criterion (1880)[16] for the pointwise convergence of Fourier series.
Any Hölder continuous function $f \in C^\alpha$ such that $|f(x-y)-f(x)| \leqslant C|y|^\alpha$ for $|y|<\delta$ for some $\delta>0$ satisfies the Dini's criterion (1.17). In particular, Fourier sine series in (1.4) converges uniformly on $[0,\pi]$ to the function $f$ in (1.5).

[Sep 15, 2026]

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[Sep 16, 2026]

**Du Bois-Reymond's counterexample** However, continuous functions need not satisfy Dini's condition, and we have the negative result by P. du Bois-Reymond (1873)[17]: *There exists a continuous function whose Fourier series diverges at a point.*
We don't follow his original proof, while apply the Banach-Steinhaus theorem (so-called uniform boundedness principle) instead to show this negative result. To this end, let[18]

$$Z = \{f \in C([-\pi,\pi];\mathbb{C}) \mid \|f\|_\infty := \sup{}_{x\in[-\pi,\pi]}|f(x)| < \infty\} = C_b([-\pi,\pi];\mathbb{C})$$

denote the usual continuous function space on $[-\pi,\pi]$, which is a Banach space. Let $\{T_N\}_N$ be a family of linear bounded operators from $Z$ to $\mathbb{C}$, which is a normed vector space, given by

$$T_N : Z \to \mathbb{C}, \text{ via } T_N f = (S_N f)(0) = \int_{-\pi}^{\pi} f(t) D_N(t)\,\mathrm{d}t\,,$$

where $D_N(x) = \frac{\sin((N+\frac{1}{2})x)}{2\pi\sin(\frac{1}{2}x)}$. By Banach-Steinhaus theorem, either

$$\sup{}_N \|T_N\|_{Z\to\mathbb{C}} < \infty,$$

or there exists an element $f \in Z$ such that

$$\sup{}_N |T_N f| = \infty.$$

[16]Ulisse Dini, *Serie di Fourier e altre rappresentazioni analitiche delle funzioni di una variabile reale.* Pisa: Tipografia T. Nistri, 1880.

[17]Paul du Bois-Reymond, Über die Fourier'schen Reihen, *Nachrichten der Königlichen Gesellschaft der Wissenschaften und der Georg-August-Universität zu Göttingen*, 1873.

[18]The space $Z$ here is different from the space $Y$ given in (1.12).

For any fixed $N \in \mathbb{N}$, $D_N$ has only a finite number of zeros, and hence $\operatorname{sgn}(D_N)$ has only a finite number of jumps. Thus for any $\varepsilon > 0$, there exists a continuous function $f$ such that

$$\|f\|_\infty = 1 \text{ and } T_N f \geqslant d_N - \varepsilon, \quad d_N := \int_{-\pi}^{\pi} |D_N(t)| \,\mathrm{d}t\,.$$

Obviously the operator norm $\|T_N\|_{Z\to\mathbb{C}}$ is bounded by $d_N$, and therefore

$$\|T_N\|_{Z\to\mathbb{C}} = d_N. \tag{1.18}$$

Recall (1.16): $d_N = O(\ln N)$ as $N \to \infty$, and hence

$$\|T_N\|_{Z\to\mathbb{C}} = O(\ln N) \text{ as } N \to \infty.$$

Banach-Steinhaus theorem then implies that there exists a continuous function $f$ such that

$$|T_N f| = |(S_N f)(0)| \to \infty \text{ as } N \to \infty.$$

To conclude, we find a continuous function $f \in C([-\pi, \pi])$, which is uniformly bounded on $[-\pi, \pi]$, while $|(S_N f)(0)| \to \infty$, and hence $(S_N f)(0) \nrightarrow f(0)$ as $N \to \infty$.

### 1.3.3 Dirichlet-Jordan's criterion

The Jordan's criterion (1881)[19] investigates this pointwise convergence property from another perspective. Namely, if *the function $f$ has a bounded variation in the neighborhood of the point $x$,* then the pointwise convergence holds in the following sense:

$$\lim_{N\to\infty} S_N f(x) = \frac{1}{2}(f(x-0) + f(x+0)). \tag{1.19}$$

We give a short proof here. Since every function of bounded variation is the difference of two monotonic functions, we assume without loss of generality that $f$ is monotonic in some neighborhood of the point $x$. Since $D_N(t) = D_N(-t)$, we rewrite

$$S_N f(x) = \int_0^{\pi} f(x-t) D_N(t) \,\mathrm{d}t + \int_{-\pi}^{0} f(x-t) D_N(t) \,\mathrm{d}t$$

[19] Camille Jordan, Sur la série de Fourier, *Comptes Rendus de l'Académie des Sciences de Paris*, 1881.

$$= \int_0^{\pi} \big(f(x-t)+f(x+t)\big) D_N(t)\,\mathrm{d}t\,,$$

and it suffices to show for any monotonic function $g$, it holds

$$\lim_{N\to\infty} \int_0^{\pi} g(t) D_N(t)\,\mathrm{d}t = \frac{1}{2} g(0_+). \tag{1.20}$$

Indeed, one may assume further $g(0_+) = 0$, then for any $\varepsilon > 0$ there exists $\delta > 0$ such that $|g||_{(0,\delta)} < \varepsilon$, and there exists $\xi \in [0, \delta]$ such that

$$\begin{aligned}\int_0^{\pi} g(t) D_N(t)\,\mathrm{d}t &= \int_0^{\delta} g(t) D_N(t)\,\mathrm{d}t + \int_{\delta}^{\pi} g(t) D_N(t)\,\mathrm{d}t \\ &= g(\delta_-) \int_{\xi}^{\delta} D_N(t)\,\mathrm{d}t + \int_{\delta}^{\pi} g(t) D_N(t)\,\mathrm{d}t\,,\end{aligned}$$

where the second equality follows from the second mean value theorem for integrals[20]. The first integral above is bounded by

$$\begin{aligned}\varepsilon \Big| \int_{\xi}^{\delta} \frac{\sin((N+\frac{1}{2})t)}{2\pi \sin(\frac{1}{2}t)}\,\mathrm{d}t \Big| &\leqslant \varepsilon \Big( \Big| \int_{\xi}^{\delta} \frac{\sin((N+\frac{1}{2})t)}{\pi t}\,\mathrm{d}t \Big| + C \Big) \\ &\leqslant \varepsilon \Big( \Big| \int_{(N+\frac{1}{2})\xi}^{(N+\frac{1}{2})\delta} \frac{\sin t}{\pi t}\,\mathrm{d}t \Big| + C \Big) \leqslant \tilde{C}\varepsilon,\end{aligned}$$

while the second integral tends to 0 as $N \to \infty$ by Riemann-Lebesgue lemma. Hence (1.20) holds and correspondingly (1.19) follows. In particular, if $f$ is continuous and has a bounded variation in a neighborhood of $x$, then

$$\lim_{N\to\infty} S_N f(x) = f(x).$$

[20]Given any monotonically increasing function $g$ and continuous function $f$ on $[a, b]$, we compute further the following Lebesgue-Stieltjes integral as

$$\int_a^b f(x)\,dg(x) = \big(g(b)-g(a)\big) \int_a^b f(x)\,d\frac{g(x)-g(a)}{g(b)-g(a)} \in \big(g(b)-g(a)\big)[\min_{[a,b]} f, \max{}_{[a,b]} f].$$

Thus there exists $\xi \in [a, b]$ such that $\int_a^b f(x)\,dg(x) = f(\xi)\big(g(b)-g(a)\big)$, which is the second mean value theorem for Stieltjes integrals. The same result holds for monotonically decreasing function $g$.

Now let $F(x) = \int_a^x f(t)\,\mathrm{d}t$, then we use integration by parts and the second mean value theorem above to derive

$$\begin{aligned}\int_a^b f(x)g(x)\,\mathrm{d}x &= \int_a^b F'(x)g(x)\,\mathrm{d}x = [F(x)g(x)]_a^b - \int_a^b F(x)\,dg(x) \\ &= F(b)g(b) - F(\xi)\big(g(b)-g(a)\big) = g(b)\int_{\xi}^b f(x)\,\mathrm{d}x + g(a)\int_a^{\xi} f(x)\,\mathrm{d}x\,.\end{aligned}$$

The classical Dirichlet's pointwise convergence criterion (1829)[21]:

- $f$ is bounded and piecewise continuous;
- $f$ has a finite number of maxima and minima,

is a special case of Jordan's criterion.

### 1.3.4 Fejér's kernel

The pointwise convergece theory for Fourier series of continuous functions was "rescued" by L. Fejér in 1900 [22]. Instead of the natural partial sum of Fourier series where Dirichlet kernel $D_N$ is involved, L. Fejér made use of the arithmetic means of partial sums (Cesàro's summation)

$$\sigma_N = \frac{1}{N+1}\sum_{k=0}^{N} S_k,$$

and the associated Fejér kernel

$$\begin{aligned}
F_N(y) &= \frac{1}{N+1}\sum_{k=0}^{N} D_k(y) \\
&= \frac{1}{N+1}\frac{1}{2\pi}\frac{\sin(\frac{1}{2}y)+\sin(\frac{3}{2}y)+\sin(\frac{5}{2}y)+\cdots+\sin((N+\frac{1}{2})yss)}{\sin(\frac{1}{2}y)} \\
&= \frac{1}{N+1}\frac{1}{2\pi}\frac{\cos 0-\cos((N+1)y)}{2(\sin(\frac{1}{2}y))^2} \\
&= \frac{1}{N+1}\frac{1}{2\pi}\Big(\frac{\sin(\frac{1}{2}(N+1)y)}{\sin(\frac{1}{2}y)}\Big)^2 \geqslant 0.
\end{aligned}$$

Note that unlike the sign-changing Dirichlet kernel $D_N$, $F_N \geqslant 0$ is nonnegative. $F_N$ satisfies also (1.15):

$$\int_{-\pi}^{\pi} F_N(y)\,\mathrm{d}y \ = 1 \text{ and } F_N(y) \leqslant \frac{1}{N+1}\frac{1}{2\pi}\frac{1}{(\sin(\frac{1}{2}\delta))^2} \text{ if } \delta \leqslant |y| \leqslant \pi,$$

and is an approximate identity. Then we have immediately that for any continuous function $f \in C([-\pi,\pi])$,

$$\lim_{N\to\infty} \sup{}_{x\in[-\pi,\pi]} |(\sigma_N f)(x) - f(x)| = 0.$$

[21] Peter Gustav Lejeune Dirichlet, Sur la convergence des séries trigonométriques qui servent à représenter une fonction arbitraire entre des limites données, *Journal für die reine und angewandte Mathematik* (Crelle's journal), 1829.

[22] Lipót Fejér, Sur les fonctions bornées et intégrables, *Comptes rendus hebdomadaires des séances de l'Académie des sciences*, 1900.

Indeed,

$$
\begin{aligned}
|(\sigma_N f)(x) - f(x)| &= \Big|\int_{-\pi}^{\pi} \big(f(x-t) - f(x)\big) F_N(t)\,\mathrm{d}t\Big| \\
&\leqslant \int_{|t|<\delta} |f(x-t) - f(x)| F_N(t)\,\mathrm{d}t \; + 2\|f\|_\infty \int_{\delta\leqslant|t|\leqslant\pi} F_N(t)\,\mathrm{d}t\,,
\end{aligned}
$$

where $\delta$ can be chosen such that the first integral be arbitrarily small, while for any fixed $\delta$, the second integral tends to 0 as $N$ tends to $\infty$.
The above argument can be used to show the $L^p(-\pi,\pi)$ convergence of $f$ by $\sigma_N f$ for $p \in [1,\infty)$, as long as one notes $\lim_{t\to 0} \|f(x-t) - f(x)\|_{L^p_x} = 0$ for $p \in [1,\infty)$. This implies the following facts:

- The trigonometric polynomials are dense in $L^p(-\pi,\pi)$, $p \in [1,\infty)$.
- If $f$ is integrable with $c_n = 0$ for all $n \in \mathbb{Z}$, then $f$ is identically zero.

## 1.4 Convergence in norm of Fourier series

We have already achieved the satisfactory convergence result of Fourier series in the sense of $L^2(-\pi,\pi)$ by the spectral theorem:

$$\lim_{N\to\infty} \|S_N f - f\|_{L^2(-\pi,\pi)} = 0, \quad \forall f \in L^2(-\pi,\pi). \tag{1.21}$$

where the partial sum $S_N f$ is defined in (1.14). Furthermore, the mapping $f \mapsto (c_n)$ is an isometry from $L^2(-\pi,\pi)$ to $\ell^2(\mathbb{Z})$, that is,

$$\|f\|^2_{L^2(-\pi,\pi)} = \sum_{n=-\infty}^{\infty} |c_n|^2. \tag{1.22}$$

We may also ask whether the convergence result (1.21) holds in the sense of $L^p(-\pi,\pi)$.

- Case $p = \infty$.

  Du Bois-Reymond's counterexample in Subsection 1.3.2 gives the negative answer for the case $p = \infty$.

- If $p \in [1,\infty)$, we have the following equivalence relation for $p \in [1,\infty)$:

  $$\|S_N f - f\|_{L^p(-\pi,\pi)} \to 0 \text{ as } N \to \infty, \quad \forall f \in L^p(-\pi,\pi) \tag{1.23}$$
  $$\Leftrightarrow \exists C = C_p \text{ such that uniformly in } N, \quad \|S_N\|_{L^p(-\pi,\pi)\to L^p(-\pi,\pi)} \leqslant C. \tag{1.24}$$

  This equivalence relation follows from the following facts:

- Density property: $C_0^\infty(-\pi,\pi)$ is dense in $L^p(-\pi,\pi)$;
- Uniform convergence and hence convergence in $L^p$-norm for a dense subset $C_0^\infty(-\pi,\pi)$: For any $\varphi \in C_0^\infty(-\pi,\pi)$, $\|S_N\varphi-\varphi\|_{L^p(-\pi,\pi)} \leqslant (2\pi)^{\frac{1}{p}}\|S_N\varphi-\varphi\|_{L^\infty(-\pi,\pi)} \to 0$ as $N \to 0$,

and the decomposition $S_N f - f = S_N(f-\varphi) + (S_N\varphi - \varphi) + (\varphi - f)$. Therefore

- Case $p = 1$.
  Similar as the analysis for the norm of $T_N$ in (1.18), we obtain

  $$\|S_N\|_{L^1\to L^1} = O(\ln N) \text{ as } N \to \infty,$$

  and thus the convergence statement (1.23) fails for $p = 1$.
- Case $p \in (1,\infty)$.
  We claim that the convergence result (1.23) for $p \in (1,\infty)$ follows from the $L^p$-boundedness of the conjugate function discussed below.

### 1.4.1 Conjugate function and Riesz projections

In the functional setting of $L^2(-\pi,\pi)$, we have shown (1.13):

$$f(x) = \frac{1}{\sqrt{2\pi}}\sum_{n=-\infty}^{\infty}\widehat{f}(n)e^{inx}, \quad \widehat{f}(n) := c_n = \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} f(x)e^{-inx}\,\mathrm{d}x\,. \quad (1.25)$$

We define the conjugate function of a function $f$ as[23]

$$\mathcal{H}(f)(x) = \frac{1}{\sqrt{2\pi}}\sum_{n=-\infty}^{\infty}(-i\,\mathrm{sgn}\,(n))\widehat{f}(n)e^{inx}, \quad (1.26)$$

where $\mathrm{sgn}\,(0) = 0$, as well as Riesz projections $P_\pm$ by [24]

$$P_+(f)(x) = \frac{1}{\sqrt{2\pi}}\sum_{n=0}^{\infty}\widehat{f}(n)e^{inx},$$

[23] We will discuss the counter part, namely the Hilbert transform, on the whole line in Subsection 3.3.2 later.

[24] Hardy space $L^2_+$ consists of $L^2$ functions with Fourier transform supported on the domain $\{n \geqslant 0\}$, that is $L^2_+ = \{P_+ f \mid f \in L^2\}$. It identifies holomorphic functions $F$ on the unit disc such that $\sup_{r<1}\int_{-\pi}^{\pi}|F(re^{i\theta})|^2 d\theta < \infty$.

$$P_-(f)(x) = -\frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{-1} \widehat{f}(n) e^{inx}.$$

Then the following relations hold

$$f = P_+(f) - P_-(f), \quad P_+(f) + P_-(f) = i\mathcal{H}(f) + \frac{1}{\sqrt{2\pi}} \widehat{f}(0),$$

or equivalently,

$$P_\pm(f) = \frac{1}{2}\Big(i\mathcal{H}(f) \pm f + \frac{1}{\sqrt{2\pi}} \widehat{f}(0)\Big).$$

We now reformulate the partial sum $S_N$ in terms of Riesz operator $P_+$ and multiplication operators $e^{iNx}$, $e^{i2Nx}$. We first rewrite

$$\begin{aligned} S_N f &= \frac{1}{\sqrt{2\pi}} \sum_{n=-N}^{N} \widehat{f}(n) e^{inx} \\ &= \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{2N} \widehat{f}(n-N) e^{i(n-N)x} \\ &= e^{-iNx} \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{2N} \widehat{f(\cdot)e^{iN\cdot}}(n) e^{inx}, \end{aligned}$$

where we compute for any $g \in L^2(-\pi,\pi)$ that

$$\begin{aligned} \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{2N} \widehat{g}(n) e^{inx} &= \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \widehat{g}(n) e^{inx} - \frac{1}{\sqrt{2\pi}} \sum_{n=2N+1}^{\infty} \widehat{g}(n) e^{inx} \\ &= P_+(g) - e^{i(2N+1)x} \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \widehat{g(\cdot)e^{-i(2N+1)\cdot}}(n) e^{inx} \\ &= P_+(g) - e^{i(2N+1)x} P_+(g e^{-i(2N+1)\cdot}). \end{aligned}$$

To conclude, we can represent the partial sum operator $S_N$ as

$$S_N f = e^{-iNx} P_+(f e^{iN\cdot}) - e^{i2Nx} P_+(f e^{-i2N\cdot}).$$

Since the multiplication operator $e^{iax}$, $a \in \mathbb{R}$ is a bounded operator on $L^p$-space with operator norm 1, we derive for all $N \in \mathbb{N}$,

$$\|S_N\|_{L^p(-\pi,\pi)\to L^p(-\pi,\pi)} \leqslant 2\|P_+\|_{L^p(-\pi,\pi)\to L^p(-\pi,\pi)},$$

which is bounded by the norm $\|\mathcal{H}\|_{L^p(-\pi,\pi)\to L^p(-\pi,\pi)}$, up to some constants.

The fact[25] that $\mathcal{H}$ is a bounded operator in $L^p(-\pi,\pi)$, $p \in (1,\infty)$ implies the uniform $L^p$-boundedness of $S_N$ in (1.24), and hence the convergence result (1.23) holds for $p \in (1,\infty)$.
Historically, the $L^p(-\pi,\pi), p \in (1,\infty)$-norm convergence of Fourier series was established by M. Riesz (1927) [26], who proved the $L^p$ boundedness of the conjugate function for $p \in (1,\infty)$. A modern proof rests on the Marcinkiewicz interpolation theorem[27] interpolating the $L^2$ isometry (1.22) against Kolmogorov's weak $(1,1)$ bound[28] yields the strong $(p,p)$ inequality for $p \in (1,2)$, while the range $p \in (2,\infty)$ follows by duality, the adjoint of the conjugation operator being its own negative.
The real-variable form of the argument is due to Calderón and Zygmund[29], whose decomposition replaced the complex-analytic proofs and supplied the weak $(1,1)$ estimate for general singular integrals in $\mathbb{R}^n$; in this formulation the decisive object is the maximal function[30], $\sup_{t>0}|Q_t f|$ for the conjugate function and $\sup_{r<1}|P_r f|$ for the Poisson kernel, whose weak $(1,1)$ and strong $(p,p)$ bounds govern pointwise and norm behaviour alike, and whose partial-sum analogue $\sup_N|S_N f|$ is precisely what later carried the problem from norm convergence to almost-everywhere convergence.

[25]We don't give the proof here, which can be found e.g. in [3, Subsection 4.1.2].

[26]Marcel Riesz, Sur les fonctions conjuguées, *Math. Z.* 27, 218–244, 1927 (announced in Les fonctions conjuguées et les séries de Fourier, *C. R. Acad. Sci. Paris* 178, 1464–1467, 1924).

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[28]Andrey Nikolaevich Kolmogorov, Sur les fonctions harmoniques conjuguées et les séries de Fourier, *Fund. Math.* 7, 23–28, 1925.

[29]Alberto Pedro Calderón and Antoni Zygmund, On the existence of certain singular integrals, *Acta Math.* 88, 85–139, 1952.

[30]Godfrey Harold Hardy and John Edensor Littlewood, A maximal theorem with function-theoretic applications, *Acta Math.* 54, 81–116, 1930.

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