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Lecture Notes for "Fourier Analysis and Differential Equations" [1] [2]

Time [3] & Place:

Tuesday, 13:30-15:05, 研教楼 306
(Sep. 8, Sep. 15, Sep. 22, Sep. 29, Oct. 13, Oct. 20, Oct. 27, Nov. 3)

Wednesday, 10:05-11:40, 综合教学 1 号楼 356
(Sep. 9, Sep. 16, Sep. 23, Sep. 30, Oct. 14, Oct. 21, Oct. 28, Nov. 4)

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[1] The lecture notes are based on a lecture given at the Dalian University of Technology during the fall term 2026.

[2] Comments are welcome to be sent to me by email.

[3] It takes place from Sep. 6 to Nov. 6, 2026 (2.-10. semester week).

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This two-month lecture provides an introduction to the sophisticated theory of Fourier analysis. The one-dimensional case is particularly emphasized, serving as motivation and providing examples for the illustration of the general theory. It also helps to connect the various concepts encountered in the theories of ordinary differential equations (ODEs), functional analysis, complex analysis, harmonic analysis and many other fields. Due to time constraints and the ambitious goal, we are unable to define all the related mathematical concepts rigorously, and instead aim to draw the audience's interest to the relevant theory. (The footnotes are left as exercises to the audience.) Finally the application to partial differential equations (PDEs) is demonstrated.

1 Fourier series

1.1 Examples of boundary value problems

1.1.1 An ODE with Dirichlet boundary condition

Consider the boundary value problem for a second-order ordinary differential equation on the interval $[0, \pi]$,

$$u''(x) + \lambda u(x) = 0 \text{ on } (0, \pi), \quad u(0) = u(\pi) = 0. \tag{1.1}$$

We consider three different cases:

- Case $\lambda < 0$.

  The following two functions are the fundamental solutions for the ODE $u''(x) = -\lambda u(x)$

  $$\sinh(\sqrt{-\lambda}x) = \frac{e^{\sqrt{-\lambda}x} - e^{-\sqrt{-\lambda}x}}{2},$$
  $$\cosh(\sqrt{-\lambda}x) = \frac{e^{\sqrt{-\lambda}x} + e^{-\sqrt{-\lambda}x}}{2},$$

  The boundary conditions imply that the solution should be zero. [4]

- Case $\lambda = 0$.

  A regular solution of $u''(x) = 0$ is a linear function $u(x) = ax + b$, which is 0 due to the boundary conditions.

[4] One can also argue from another perspective: We take the $L^2(0, \pi)$-inner product

- Case $\lambda > 0$.

  The following two functions are fundamental solutions for the ODE $u''(x) = -\lambda u(x)$

$$\sin(\sqrt{\lambda}x) = \frac{e^{i\sqrt{\lambda}x} - e^{-i\sqrt{\lambda}x}}{2i},$$
$$\cos(\sqrt{\lambda}x) = \frac{e^{i\sqrt{\lambda}x} + e^{-i\sqrt{\lambda}x}}{2},$$

  The boundary conditions imply that the solution should be a linear combination of

$$\sin(nx), \quad n = 1, 2, \cdots$$

To conclude, there is an infinite set of discrete values of λ:

$$\lambda = n^2, \quad n = 1, 2, \cdots$$

such that the boundary value problem (1.1) has a nontrivial solution. These values n^2, $n \in \mathbb{N}$ are called the eigenvalues of the (eigenvalue) problem (1.1) and the functions

$$u_n(x) = b_n \sin(nx), \quad n \in \mathbb{N}, \quad b_n \in \mathbb{R}(\text{or } \mathbb{C})$$

are the corresponding eigenfunctions. The concept of eigenvalues and eigenfunctions becomes clearer in Section 1.2 below.

1.1.2 Fourier's idea to solve the heat equation

Consider the one-dimensional initial-boundary value problem for the (real-valued) heat equation

$$\begin{cases} \partial_t u - \partial_{xx} u = 0 \text{ on } (0,\infty) \times (0,\pi), \\ u(0,x) = f(x) \quad x \in (0,\pi), \\ u(t,0) = u(t,\pi) = 0 \quad t \in (0,\infty), \end{cases} \tag{1.3}$$

between the equation and the solution u itself, to obtain by integration by parts

$$-\int_0^\pi |u'|^2 \, \mathrm{d}x + \lambda \int_0^\pi |u|^2 \, \mathrm{d}x = 0. \tag{1.2}$$

This implies, if u is a regular solution (e.g. in $H^1(0,\pi)$), then $u = 0$. This argument can be easily generalized to the case when $\lambda \in \mathbb{C}$ with $\operatorname{Im}\lambda \neq 0$, where $u = 0$ (in $H^1(0,\pi)$) follows.

where the compatibility condition is satisfied

$$f(0) = f(\pi) = 0,$$

and we assume that f is not identically zero: $f \neq 0$.
J. Fourier[5] made the following Ansatz (separation of variables)

$$u(t, x) = T(t)X(x).$$

This leads to the following:

- $T'(t)X(x) - T(t)X''(x) = 0$, which implies (at least formally),

$$\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = -\lambda \text{ for some constant } \lambda.$$

- $T(t)X(0) = T(t)X(\pi) = 0$ for all time t, which implies

$$X(0) = X(\pi) = 0.$$

- $T(0)X(x) = f(x)$.

From the ODE results above, if $f \neq 0$ and one searches for nontrivial (real-valued) solution, λ can "only" take the value n^2 for some $n \in \mathbb{N}$, and $X(x)$ is a multiple of $\sin(nx)$, and in this case, the time function $T(t) = T(0)e^{-n^2 t}$. That is, if the initial data is

$$f(x) = b_n \sin(nx),$$

for some $n \in \mathbb{N}$ and some constant b_n, then the problem (1.3) has a (unique) solution

$$u(t, x) = T(t)X(x) = T(0)e^{-n^2 t}\frac{b_n \sin(nx)}{T(0)} = b_n e^{-n^2 t}\sin(nx).$$

If the initial data is a finite sum of sine functions

$$f(x) = \sum_{n=1}^{N} b_n \sin(nx),$$

then by the superposition principle the solution is

$$u(t, x) = \sum_{n=1}^{N} b_n e^{-n^2 t}\sin(nx).$$

[5]Joseph Fourier, *Théorie Analytique de la Chaleur*. Chez F. Didot, 1822 (Firstly presented to *Institute de France* in 1807).

Thus, formally, if we can expand the initial data $f(x)$ as a sine series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx), \tag{1.4}$$

then the following function (when it is well-defined) is a solution of (1.1)

$$u(x) = \sum_{n=1}^{\infty} b_n e^{-n^2 t} \sin(nx).$$

A function with such an expansion (1.4) (nowadays called Fourier sine series) "seems" very special. For half a century around 1800 there was a great debate whether such a series expansion holds only for specific functions.

1.2 Spectral theorem and Fourier series

We claim that any function in the functional space

$$\{f \in C^2([0,\pi]) \,|\, f(0) = f(\pi) = 0\} \tag{1.5}$$

can be expanded as Fourier sine series. Historically one can make use of the Weierstrass approximation theorem and the density of trigonometric functions in the space of polynomial functions to prove this statement. Here we illustrate this in modern mathematical language.
We can reformulate (1.1) as the following eigenvalue problem

$$\mathcal{A}u = \lambda u, \tag{1.6}$$

where the operator $\mathcal{A}$ is defined by[6]

$$\mathcal{A} : D(\mathcal{A}) \to X, \quad \mathcal{A}(u) = -u'' \text{ for } u \in D(\mathcal{A}). \tag{1.7}$$

Here we take the underlying inner product space and scalar product below

$$X = C([0,\pi];\mathbb{C}), \quad \langle u, v\rangle = \int_0^\pi u\bar{v}\,\mathrm{d}x\,,$$

and the domain of the operator $\mathcal{A}$ is a subspace of X:

$$D(\mathcal{A}) = \{u \in C^2([0,\pi]) \,|\, u(0) = u(\pi) = 0\} \subset X.$$

[6]There is a certain conceptual mismatch between the classical setting of continuous functions and the spectral-theoretic framework, which is more naturally aligned with spaces such as $L^2(0,\pi)$. For example, one may define $\mathcal{A} : H_0^2(0,\pi) = D(\mathcal{A}) \to L^2(0,\pi)$.

The induced norm on X is

$$\|u\| = \langle u, u\rangle^{\frac{1}{2}},$$

and the distance between u and v is $\|u-v\|$. Note that $D(\mathcal{A})$ and X are not complete with respect to the defined metric.
Recall that the Lebesgue space[7]

$$L^2(0,\pi) = \{f \text{ is Lebesgue measurable on } (0,\pi) \mid \|f\|_{L^2(0,\pi)} = \Big(\int_0^\pi |f(x)|^2 \,\mathrm{d}x\Big)^{\frac{1}{2}} < \infty\}/\sim,$$

where

- The integral is understood in the sense of Riemann-Lebesgue:

$$\|f\|_{L^2(0,\pi)} = \Big(2\int_0^\infty \alpha\; m(\{x \in (0,\pi) \mid |f(x)| > \alpha\})\, d\alpha\Big)^{\frac{1}{2}}$$

- The equivalence $\sim$ means identity almost everywhere (a.e.):

$$f \sim g \Leftrightarrow m(\{x \in (0,\pi) \mid f(x) \neq g(x)\}) = 0.$$

The Lebesgue space $L^2(0,\pi)$ is endowed with the scalar product $\langle\cdot,\cdot\rangle$, and is a Hilbert space. By the density of the space[8] of test functions $C_c^\infty(0,\pi)$ in $L^2(0,\pi)$, the closure of $C_c^\infty(0,\pi) \subset D(A) \subset X$ in the Lebesgue space $(L^2(0,\pi), \langle\cdot,\cdot\rangle)$ is the Lebesgue space itself

$$\overline{X}^{L^2} = \overline{D(\mathcal{A})}^{L^2} = \overline{C_c^\infty(0,\pi)}^{L^2} = L^2(0,\pi).$$

1.2.1 Resolvent computation

From the ODE results above (recalling (1.2)), the eigenvalues and normalized eigenfunctions of the operator $\mathcal{A}$ are

$$\lambda_n = n^2, \quad u_n = \sqrt{\frac{2}{\pi}}\sin(nx), \quad n \in \mathbb{N}, \tag{1.8}$$

with $\|u_n\|_{L^2(0,\pi)} = 1$.

[Sep 8, 2026]

[7] See Subsection 1.2 for more detail.
[8] The density result also holds on the whole line or in higher dimensions.

We now compute the resolvent of $\mathcal{A}$ at $\lambda = -1 \notin \mathbb{N}$, that is, we compute $R_{\mathcal{A}}(-1) = (\mathcal{A}+1)^{-1} : X \to D(\mathcal{A})$. That is, for a given function $g \in X$, we seek the solution $u \in D(\mathcal{A})$ of the following boundary value problem for the inhomogeneous ODE:

$$-u'' + u = g, \quad u(0) = u(\pi) = 0. \tag{1.9}$$

It suffices to compute the Green function of this boundary value problem. The homogeneous equation $-u'' + u = 0$ has fundamental solutions

$$e^{-x}, \quad e^{x}.$$

The solution satisfying the boundary condition $u(0) = 0$ is a multiple of

$$\phi(x) = e^{x} - e^{-x},$$

while the solution satisfying the boundary condition $u(\pi) = 0$ is a multiple of

$$\psi(x) = e^{-(\pi - x)} - e^{\pi - x}.$$

Their Wronskian $W(\phi, \psi) = \phi\psi' - \phi'\psi = \det \begin{pmatrix} \phi & \psi \\ \phi' & \psi' \end{pmatrix}$ is independent of x [9] and is given by

$$W(\phi, \psi) = 2(e^{\pi} - e^{-\pi}).$$

The Green function[10] is

$$\begin{aligned} G(x,y) &= \begin{cases} -\frac{\phi(x)\psi(y)}{W(\phi,\psi)} & \text{for } x < y \\ \\ -\frac{\phi(y)\psi(x)}{W(\phi,\psi)} & \text{for } x > y \end{cases} \\ &= -\frac{e^{-\pi+x+y} + e^{\pi-(x+y)} - e^{\pi-|x-y|} - e^{-\pi+|x-y|}}{W(\phi,\psi)}. \end{aligned}$$

[9] It follows from Abel's identity (also called Liouville's formular).

[10] The $-$ sign before the quantity $\frac{\phi(x)\psi(y)}{W(\phi,\psi)}$ or $\frac{\phi(y)\psi(x)}{W(\phi,\psi)}$ in the definition of the Green function here is due to the $-$ sign before the highest order derivative ∂_{xx} in the equation (1.9), which (usually) is assumed to be $+$ in the standard formulation of ODE.

Thus the unique solution of (1.9) reads[11]

$$u(x) = \int_0^\pi G(x,y)g(y)\,\mathrm{d}y\,, \tag{1.10}$$

that is,

$$R_{\mathcal{A}}(-1)g(x) = (\mathcal{A}+1)^{-1}g(x) = \int_0^\pi G(x,y)g(y)\,\mathrm{d}y\,.$$

One may verify that

- $R_{\mathcal{A}}(-1) : X \to D(\mathcal{A})$.

  This follows immediately from (1.9).

- $R_{\mathcal{A}}(-1)$ is a compact operator in X. That is, for any bounded sequence $(g_k)_k$ in X, the function sequence $(R_{\mathcal{A}}(-1)g_k)_k$ has a subsequence which converges in X.

  This follows from the properties of G and a compactness argument, e.g. Arzela-Ascoli theorem.

- $R_{\mathcal{A}}(-1)$ is symmetric. That is,

  $$\big\langle R_{\mathcal{A}}(-1)g, h\big\rangle = \big\langle g, R_{\mathcal{A}}(-1)h\big\rangle, \quad \forall g, h \in X.$$

  This follows from the symmetry property of $G(x,y) = G(y,x) \in \mathbb{R}$.

1.2.2 Spectral theorem and Fourier sine series

Spectral theorem[12] states that given an infinte-dimensional inner product space X and a compact symmetric operator $\mathcal{R} : X \to X$, there exists a sequence of real eigenvalues $(z_n)_n \subset \mathbb{R}$ converging to 0. The corresponding

[11] The unique solvability of the inhomogeneous problem (1.9) follows from the fact that the homogeneous problem (1.1) with $\lambda = -1$ has only trivial zero solution, by the Fredholm alternative theorem.

The unique solution of (1.9) can be represented by Green function (1.10): $u(x) = \int_0^x G(x,y)g(y)\,\mathrm{d}y + \int_x^\pi G(x,y)g(y)\,\mathrm{d}y$, where the Green function has the following properties

- $G(x,y) \in C([0,\pi]^2)$, $G(x,y) \in C^2(\Delta_\pm)$ with $\Delta_\pm = \{(x,y) \in [0,\pi]^2 \mid \pm(x-y) \leqslant 0\}$, and $\partial_x G$ has a jump at the diagonal line: $\partial_x G(x,x-0) - \partial_x G(x,x+0) = -1$;
- $G(\cdot, y)$ satisfies the homogeneous ODE (with respect to x variable) in $\Delta_\pm$ and the boundary conditions.

[12] We don't give the proof here.

normalized eigenvectors $(v_n)_n$ form an orthonormal set and every $f \in \mathrm{Ran}(\mathcal{R})$ can be written as

$$f = \sum_{n=1}^{\infty} \langle f, v_n \rangle v_n.$$

If $\mathrm{Ran}(\mathcal{R})$ is dense, then the eigenvectors form an orthonormal basis.
Thus for the compact symmetric operator $R_{\mathcal{A}}(-1)$ in X, there exists a sequence of real eigenvalues $(z_n) \to 0$ and the corresponding normalized eigenfunctions

$$(v_n) \subset D(\mathcal{A}), \quad \|v_n\| = \langle v_n, v_n \rangle = 1,$$

which form an orthonormal basis in the closure of its range $\overline{\mathrm{Ran}(R_{\mathcal{A}}(-1))}^{L^2} = \overline{D(\mathcal{A})}^{L^2} = L^2(0,\pi)$. That is, for any $f \in L^2(0,\pi)$, we can expand it as follows:

$$f = \sum_{n=1}^{\infty} \langle f, v_n \rangle v_n.$$

Now we return to the operator $\mathcal{A}$. The fact that $R_{\mathcal{A}}(-1)v_n = z_n v_n$ is equivalent to

$$\mathcal{A}v_n = (-1 + \frac{1}{z_n})v_n,$$

that is, v_n is a normalized eigenfunction of $\mathcal{A}$ associated with the eigenvalue $-1 + \frac{1}{z_n}$ which tends to infinity. Recalling (1.8), we find

$$\lambda_n = n^2 = -1 + \frac{1}{z_n}, \quad u_n = \sqrt{\frac{2}{\pi}} \sin(nx) = v_n.$$

To conclude, the expansion $f = \sum_{n=1}^{\infty} \langle f, v_n \rangle v_n$ for any function $f \in L^2(0,\pi)$ is indeed (1.4):

$$f = \sum_{n=1}^{\infty} b_n \sin(nx), \quad b_n = \sqrt{\frac{2}{\pi}} \langle f, v_n \rangle = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx)\, \mathrm{d}x\,.$$

This is the Fourier sine series. Let us emphasize here that the series is converges with respect to our scalar product, i.e. in the sense of $L^2(0,\pi)$. We remark that if the function $f \in D(\mathcal{A})$, then the sine series converges uniformly to f. The convergence of Fourier series is discussed in more detail in Sections 1.3 and 1.4 later.

[Sep 9, 2026]
[Sep 15, 2026]

1.2.3 Eigenvalue problem with Neumann boundary condition

Instead of Dirichlet boundary condition studied in Section 1.2, we consider the Laplace eigenvalue problem associated to the Neumann boundary condition. More precisely, we consider the eigenvalue problem

$$\mathcal{B}u = \lambda u, \quad \mathcal{B} : D(\mathcal{B}) \to X, \quad \mathcal{B}(u) = -u'' \text{ for } u \in D(\mathcal{B}), \tag{1.11}$$

where the domain of the operator $\mathcal{B}$ is

$$D(\mathcal{B}) = \{u \in C^2([0,\pi]) \,|\, u'(0) = u'(\pi) = 0\} \subset X,$$

with $X = C([0,\pi];\mathbb{C})$ equipped with the inner product $\langle \cdot, \cdot \rangle$.
The same analysis as in Subsection 1.1.1 yields the following eigenvalues and normalized eigenfunctions

$$\lambda_0 = 0, \quad \lambda_n = n^2, \quad u_0 = \sqrt{\frac{1}{\pi}}, \quad u_n = \sqrt{\frac{2}{\pi}}\cos(nx), \quad n \in \mathbb{N}\,.$$

We can then compute the resolvent $R_{\mathcal{B}}(-1) : X \to D(\mathcal{B})$ in terms of Green function

$$R_{\mathcal{B}}(-1)h(x) = \int_0^\pi G_B(x,y)h(y)\,\mathrm{d}y\,, \quad G_{\mathcal{B}}(x,y) = \begin{cases} -\frac{\phi_{\mathcal{B}}(x)\psi_{\mathcal{B}}(y)}{W(\phi_{\mathcal{B}},\psi_{\mathcal{B}})} & \text{for } x \leqslant y \\ -\frac{\phi_{\mathcal{B}}(y)\psi_{\mathcal{B}}(x)}{W(\phi_{\mathcal{B}},\psi_{\mathcal{B}})} & \text{for } x \geqslant y \end{cases}$$

and verify that it is compact and symmetric, as in Subsection 1.2.1[13]. Finally the spectral theorem ensures that any function $f \in L^2(0,\pi)$ can be expanded as Fourier cosine series

$$f = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx),$$
$$a_0 = \frac{1}{\pi}\int_0^\pi f(x)\,\mathrm{d}x\,, \quad a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos(nx)\,\mathrm{d}x\,, \quad n \in \mathbb{N},$$

in the sense of $L^2(0,\pi)$.

1.2.4 Full Fourier series

We now consider the eigenvalue problem

$$\mathcal{C}u = \lambda u, \quad \mathcal{C} : D(\mathcal{C}) \to Y, \quad \mathcal{C}(u) = -u'' \text{ for } u \in D(\mathcal{C}), \tag{1.12}$$

where

[13] We omit the details here.

- $Y = C([-\pi, \pi]; \mathbb{C})$ is equipped with the inner product $\langle \cdot, \cdot \rangle_{L^2(-\pi,\pi)}$;
- we assume periodic boundary conditions on the interval $[-\pi, \pi]$

$$D(\mathcal{C}) = \{u \in C^2([-\pi, \pi]) \,|\, u(-\pi) = u(\pi), \quad u'(-\pi) = u'(\pi)\} \subset Y.$$

There are eigenvalues

$$\lambda_0 = 0, \quad \lambda_n = n^2, \quad n \in \mathbb{N},$$

and the corresponding normalized eigenfunctions

$$u_0 = \sqrt{\frac{1}{2\pi}}, \quad \{u_n, v_n\} = \left\{\sqrt{\frac{1}{\pi}}\cos(nx),\ \sqrt{\frac{1}{\pi}}\sin(nx)\right\}, \quad n \in \mathbb{N}\,.$$

The spectral theorem implies that any function $f \in L^2(-\pi, \pi)$ can be expanded as Fourier series in the sense of $L^2(-\pi, \pi)$:

$$f = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx), \quad a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x\,,$$
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,\mathrm{d}x\,, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,\mathrm{d}x\,, \quad n \in \mathbb{N}\,.$$

Observe that

- If $f \in C([-\pi, \pi])$ is an odd function, then $a_0 = a_n = 0$, and hence the full Fourier series of any odd function on $(-\pi, \pi)$ coincides with Fourier sine series on $(0, \pi)$ in the sense of $L^2(0, \pi)$.

  Conversely, if $f \in C([0, \pi])$ with Dirichlet boundary condition $f(0) = f(\pi) = 0$, then one can extend it to an odd function in $C([-\pi, \pi])$ satisfying $f(-\pi) = f(0) = f(\pi) = 0$, and the Fourier sine series coincides with the full Fourier series on $(0, \pi)$ in the sense of $L^2(0, \pi)$.

  This is the physical intuition why the Fourier sine series is "complete" on $[0, \pi]$.

- If $f \in C([-\pi, \pi])$ is an even function, then $b_n = 0$, and hence the full Fourier series of any even function on $(-\pi, \pi)$ coincides with Fourier cosine series on $(0, \pi)$ in the sense of $L^2(0, \pi)$.

  Conversely, if $f \in C([0, \pi])$ with Neumann boundary condition $f'(0) = f'(\pi) = 0$, then one can extend it to an even function in $C([-\pi, \pi])$ satisfying $f'(-\pi) = f'(0) = f'(\pi) = 0$, and the Fourier cosine series coincides with full Fourier series on $(0, \pi)$ in the sense of $L^2(0, \pi)$.

- Replacing

$$\cos(nx) = \frac{1}{2}(e^{inx} + e^{-inx}), \quad \sin(nx) = \frac{1}{2i}(e^{inx} - e^{-inx}),$$

we can rewrite the full Fourier series in the elegant complex form

$$\begin{aligned} f &= \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} c_n e^{inx}, \quad \frac{1}{\sqrt{2\pi}} c_0 = a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-i0x} \,\mathrm{d}x\,, \\ \frac{1}{\sqrt{2\pi}} c_n &= \frac{1}{2} a_n + \frac{1}{2i} b_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} \,\mathrm{d}x\,, \quad n \in \mathbb{N}, \\ \frac{1}{\sqrt{2\pi}} c_{-n} &= \frac{1}{2} a_n - \frac{1}{2i} b_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-i(-n)x} \,\mathrm{d}x\,, \quad n \in \mathbb{N}\,. \end{aligned}$$

1.3 Pointwise convergence of Fourier series

The spectral theorem guarantees the convergence of Fourier series

$$f(x) = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} c_n e^{inx}, \quad c_n = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(x) e^{-inx} \,\mathrm{d}x \tag{1.13}$$

in the sense of $L^2(-\pi, \pi)$. There is a long history of mathematicians searching for the convergence of the partial sum of the Fourier series

$$S_N f(x) = \frac{1}{\sqrt{2\pi}} \sum_{n=-N}^{N} c_n e^{inx}, \quad c_n = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(x) e^{-inx} \,\mathrm{d}x \tag{1.14}$$

as $N \to \infty$, at first in the sense of pointwise convergence.

1.3.1 Dirichlet kernel

The first positive results is due to P.G.L. Dirichlet (1829). He rewrote the partial sum of a 2π-periodic function as

$$\begin{aligned} S_N f(x) &= \frac{1}{2\pi} \sum_{n=-N}^{N} \int_{-\pi}^{\pi} f(t) e^{-int} \,\mathrm{d}t\, e^{inx} \\ &= \frac{1}{2\pi} \sum_{n=-N}^{N} \int_{-\pi}^{\pi} f(t) e^{in(x-t)} \,\mathrm{d}t \\ &= \int_{-\pi}^{\pi} f(t) D_N(x-t) \,\mathrm{d}t \; = \int_{-\pi}^{\pi} f(x-t) D_N(t) \,\mathrm{d}t \; = (f * D_N)(x), \end{aligned}$$

where D_N is the so-called Dirichlet kernel

$$\begin{aligned} D_N(y) &= \frac{1}{2\pi}\sum_{n=-N}^{N} e^{iny} \\ &= \frac{1}{2\pi}\big(1+2\cos y+2\cos(2y)+\cdots+2\cos(Ny)\big) \\ &= \frac{1}{2\pi}\frac{1}{\sin(\frac{1}{2}y)}\Big(\sin(\frac{1}{2}y)+(\sin(\frac{3}{2}y)-\sin(\frac{1}{2}y))+(\sin(\frac{5}{2}y)-\sin(\frac{3}{2}y))+ \\ &\qquad +\cdots+\big(\sin((N+\frac{1}{2})y)-\sin((N-\frac{1}{2})y)\big)\Big) \\ &= \frac{\sin((N+\frac{1}{2})y)}{2\pi\sin(\frac{1}{2}y)}. \end{aligned} \tag{1.15}$$

Observe that $D_N(y)$ satisfies

$$\int_{-\pi}^{\pi} D_N(y)\,\mathrm{d}y \;=1 \text{ and } |D_N(y)| \leqslant \frac{1}{2\pi\sin(\frac{1}{2}\delta)} \text{ if } \delta \leqslant |y| \leqslant \pi. \tag{1.16}$$

Notice that $\int_{-\pi}^{\pi}|D_N(y)|\,\mathrm{d}y$ is growing in terms of N of the rate lnN[14]:

$$\begin{aligned} &\int_{-\pi}^{\pi}|D_N(y)|\,\mathrm{d}y \;=\frac{1}{2\pi}\int_{-\pi}^{\pi}|\frac{\sin((N+\frac{1}{2})y)}{\sin(\frac{1}{2}y)}|\,\mathrm{d}y \;=\frac{1}{\pi}\int_0^{\pi}\frac{|\sin((N+\frac{1}{2})y)|}{\frac{1}{2}y}\,\mathrm{d}y \;+O(1) \\ &=\frac{2}{\pi}\int_0^{(N+\frac{1}{2})\pi}\frac{|\sin t|}{t}\,\mathrm{d}t \;+O(1)=\frac{2}{\pi}\sum_{k=0}^{N-1}\int_{k\pi}^{(k+1)\pi}\frac{|\sin t|}{t}\,\mathrm{d}t \;+O(1) \\ &=\frac{2}{\pi}\sum_{k=0}^{N-1}\int_0^{\pi}\frac{|\sin t|}{t+k\pi}\,\mathrm{d}t \;+O(1) \\ &=\frac{2}{\pi}\int_0^{\pi}\sin t\Big(\sum_{k=0}^{N-1}\frac{1}{t+k\pi}\Big)\,\mathrm{d}t \;+O(1)=O(\ln N). \end{aligned} \tag{1.17}$$

We claim that the pointwise convergence of a Fourier series at some point x depends on the regularity/smoothness of the function at x, and hence is (surprisingly) a <u>local</u> property (cf. Riemann's "second theorem" in his Habilitation thesis (1854)[15]), although the definition of Fourier series depends on the values of this function on the whole interval $[-\pi,\pi]$. To see this,

[14]This growth is the reason for the failure of the pointwise convergence of Fourier series for continuous functions, see du Bois-Reymond's negative result later.

[15]Bernhard Riemann, Ueber die Darstellbarkeit einer Function durch eine trigonometrische Reihe. Abhandlungen der Königlichen Gesellschaft der Wissenschaften zu Göttingen 13, 87–132, 1867. (Habilitationsschrift, Göttingen, 1854)

consider first the special case $f \in L^2(-\pi,\pi)$ which satisfies $f = 0$ on a neighborhood of x: $(x-\delta, x+\delta)$, and we compute

$$\begin{aligned} S_N f(x) &= \int_{\delta\leqslant|y|\leqslant\pi} f(x-y)\frac{\sin((N+\frac12)y)}{2\pi\sin(\frac12 y)}\,\mathrm{d}y \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-y)\frac{1}{\sin(\frac12 y)}1_{\{\delta\leqslant|y|\leqslant\pi\}}\big(\frac{1}{2i}(e^{(N+\frac12)iy}-e^{-(N+\frac12)iy})\big)\,\mathrm{d}y \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\Big(f(x-y)\frac{1}{\sin(\frac12 y)}1_{\{\delta\leqslant|y|\leqslant\pi\}}\frac{1}{2i}e^{\frac12 iy}\Big)e^{iNy}\,\mathrm{d}y \\ &\quad -\frac{1}{2\pi}\int_{-\pi}^{\pi}\Big(f(x-y)\frac{1}{\sin(\frac12 y)}1_{\{\delta\leqslant|y|\leqslant\pi\}}\frac{1}{2i}e^{-\frac12 iy}\Big)e^{-iNy}\,\mathrm{d}y\,. \end{aligned}$$

This is the difference of the Fourier coefficients of two integrable functions at $-N$ and N, both of which tends to 0 as $N\to\infty$: At this point x, by the Riemann-Lebesgue lemma[16] [17],

$$\lim_{N\to\infty} S_N f(x) = 0 = f(x).$$

1.3.2 Dini's criterion and du Bois-Reymond's counterexample

Now we consider the function f with general behavior on $(x-\delta,x+\delta)$ and compute (due to the unit size of the integral of D_N)

$$S_N f(x) - f(x) = \int_{-\pi}^{\pi}\big(f(x-y)-f(x)\big)\frac{\sin((N+\frac12)y)}{2\pi\sin(\frac12 y)}\,\mathrm{d}y = \int_{|y|<\delta} + \int_{\delta\leqslant|y|\leqslant\pi}.$$

[16] For any integrable function g which is periodic on $\mathbb{R}$ with period 2π, we compute

$$\begin{aligned} \hat g(N) :&= \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} g(y)e^{-iNy}\,\mathrm{d}y = -\frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} g(y)e^{-iN(y+\frac{\pi}{N})}\,\mathrm{d}y \\ &= -\frac{1}{\sqrt{2\pi}}\int_{-\pi+\frac{\pi}{N}}^{\pi+\frac{\pi}{N}} g(t-\frac{\pi}{N})e^{-iNt}\,\mathrm{d}t = -\frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} g(y-\frac{\pi}{N})e^{-iNy}\,\mathrm{d}y\,, \end{aligned}$$

and thus

$$\hat g(N) = \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} g(y)e^{-iNy}\,\mathrm{d}y = \frac12\frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi}\big(g(y)-g(y-\frac{\pi}{N})\big)e^{-iNy}\,\mathrm{d}y\,. \tag{1.18}$$

If g is continuous, then its Fourier coefficient $\hat g(N)$ tends to 0 as N tends to infinity. If $g\in L^1(-\pi,\pi)$, then we can approximate it with some continuous function to derive the convergence result.

[17] Henri Lebesgue, *Leçons sur les séries trigonométriques*, Gauthier-Villars, Paris, 1906; the integral itself is from *Intégrale, longueur, aire*, *Annali di Matematica Pura ed Applicata* (3) 7, 231–359, 1902.

Dini's criterion If we assume the following continuity condition for the function f at the point x:

$$\int_{|y|<\delta} \Big| \frac{f(x-y)-f(x)}{y} \Big| \, \mathrm{d}y \, < \infty, \tag{1.19}$$

for some $\delta > 0$, then all the functions

$$\frac{f(x-y)-f(x)}{\sin(\frac{1}{2}y)} 1_{\{|y|<\delta\}} \frac{1}{2i} e^{\pm\frac{1}{2}iy}, \quad \frac{f(x-y)-f(x)}{\sin(\frac{1}{2}y)} 1_{\{\delta\leqslant|y|\leqslant\pi\}} \frac{1}{2i} e^{\pm\frac{1}{2}iy}$$

are integrable on $[-\pi, \pi]$, and hence Riemann-Lebesgue lemma implies

$$S_N f(x) \to f(x) \text{ as } N \to \infty.$$

The continuity property (1.19) is called the Dini's criterion (1880)[18] for the pointwise convergence of Fourier series.
Any Hölder continuous function $f \in C^\alpha$ such that $|f(x-y)-f(x)| \leqslant C|y|^\alpha$ for $|y| < \delta$ for some $\delta > 0$ satisfies the Dini's criterion (1.19). In particular, Fourier sine series in (1.4) converges uniformly on $[0, \pi]$ to the function f in (1.5).

[Sep 15, 2026]
[Sep 16, 2026]

Du Bois-Reymond's counterexample However, continuous functions need not satisfy Dini's condition, and we have the negative result by P. du Bois-Reymond (1873)[19]: *There exists a continuous function whose Fourier series diverges at a point.*
We don't follow his original proof, while apply the Banach-Steinhaus theorem (so-called uniform boundedness principle) instead to show this negative result. To this end, let[20]

$$Z = \{f \in C([-\pi,\pi];\mathbb{C}) \,|\, \|f\|_\infty = \sup{}_{x\in[-\pi,\pi]}|f(x)| < \infty\} = C_b([-\pi,\pi];\mathbb{C})$$

denote the usual continuous function space on $[-\pi, \pi]$, which is a Banach space. Let $\{T_N\}_N$ be a family of linear bounded operators from Z to $\mathbb{C}$, which is a normed vector space, given by

$$T_N : Z \to \mathbb{C}, \text{ via } T_N f = (S_N f)(0) = \int_{-\pi}^{\pi} f(t) D_N(t) \, \mathrm{d}t \,,$$

[18] Ulisse Dini, *Serie di Fourier e altre rappresentazioni analitiche delle funzioni di una variabile reale*. Pisa: Tipografia T. Nistri, 1880.

[19] Paul du Bois-Reymond, Über die Fourier'schen Reihen, *Nachrichten der Königlichen Gesellschaft der Wissenschaften und der Georg-August-Universität zu Göttingen*, 1873.

[20] The space Z here is different from the space Y given in (1.12).

where $D_N(x) = \frac{\sin((N+\frac{1}{2})x)}{2\pi \sin(\frac{1}{2}x)}$. By Banach-Steinhaus theorem[21], either

$$\sup_N \|T_N\|_{Z\to\mathbb{C}} < \infty,$$

or there exists an element $f \in Z$ such that

$$\sup_N |T_N f| = \infty.$$

For any fixed $N \in \mathbb{N}$, D_N has only a finite number of zeros, and hence $\operatorname{sgn}(D_N)$ has only a finite number of jumps. Thus for any $\varepsilon > 0$, there exists a continuous function f such that

$$\|f\|_\infty = 1 \text{ and } T_N f \geqslant d_N - \varepsilon, \quad d_N := \int_{-\pi}^{\pi} |D_N(t)|\,\mathrm{d}t\,.$$

Obviously the operator norm $\|T_N\|_{Z\to\mathbb{C}}$ is bounded by d_N, and therefore

$$\|T_N\|_{Z\to\mathbb{C}} = d_N. \tag{1.20}$$

Recall (1.17): $d_N = O(\ln N)$ as $N \to \infty$, and hence

$$\|T_N\|_{Z\to\mathbb{C}} = O(\ln N) \text{ as } N \to \infty.$$

Banach-Steinhaus theorem then implies that there exists a continuous function f such that

$$\limsup_{N\to\infty} |(S_N f)(0)| = \infty.$$

To conclude, we find a continuous function $f \in C([-\pi,\pi])$, which is uniformly bounded on $[-\pi,\pi]$, while $|(S_N f)(0)| \to \infty$, and hence $(S_N f)(0) \nrightarrow f(0)$ as $N \to \infty$.

1.3.3 Dirichlet-Jordan's criterion

The Jordan's criterion (1881)[22] investigates this pointwise convergence property from another perspective. Namely, if *the function f has a bounded variation in the neighborhood of the point x,* then the pointwise convergence holds in the following sense:

$$\lim_{N\to\infty} S_N f(x) = \frac{1}{2}(f(x-0) + f(x+0)). \tag{1.21}$$

[21]Stefan Banach and Hugo Dyonizy Steinhaus, *Sur le principe de la condensation de singularités*, Fundamenta Mathematicae 9, 50–61, 1927.

[22]Camille Jordan, Sur la série de Fourier, *Comptes Rendus de l'Académie des Sciences de Paris*, 1881.

We give a short proof here. Since every function of bounded variation is the difference of two monotonic functions, we assume without loss of generality that f is monotonic in some neighborhood of the point x. Since $D_N(t) = D_N(-t)$, we rewrite

$$\begin{aligned} S_N f(x) &= \int_0^\pi f(x-t)D_N(t)\,\mathrm{d}t \,+ \int_{-\pi}^0 f(x-t)D_N(t)\,\mathrm{d}t \\ &= \int_0^\pi \big(f(x-t)+f(x+t)\big)D_N(t)\,\mathrm{d}t\,, \end{aligned}$$

and it suffices to show for any monotonic function g, it holds

$$\lim_{N\to\infty} \int_0^\pi g(t)D_N(t)\,\mathrm{d}t \,= \frac{1}{2}g(0_+). \tag{1.22}$$

Indeed, one may assume further $g(0_+) = 0$, then for any $\varepsilon > 0$ there exists $\delta > 0$ such that $|g||_{(0,\delta)} < \varepsilon$, and there exists $\xi \in [0,\delta]$ such that

$$\begin{aligned} \int_0^\pi g(t)D_N(t)\,\mathrm{d}t \,&= \int_0^\delta g(t)D_N(t)\,\mathrm{d}t \,+ \int_\delta^\pi g(t)D_N(t)\,\mathrm{d}t \\ &= g(\delta_-)\int_\xi^\delta D_N(t)\,\mathrm{d}t \,+ \int_\delta^\pi g(t)D_N(t)\,\mathrm{d}t\,, \end{aligned}$$

where the second equality follows from the second mean value theorem for integrals[23]. The first integral above is bounded by

$$\varepsilon\Big|\int_\xi^\delta \frac{\sin((N+\frac{1}{2})t)}{2\pi\sin(\frac{1}{2}t)}\,\mathrm{d}t\,\Big| \leqslant \varepsilon\Big(\Big|\int_\xi^\delta \frac{\sin((N+\frac{1}{2})t)}{\pi t}\,\mathrm{d}t\,\Big| + C\Big)$$

[23]Given any monotonically increasing function g and continuous function f on $[a,b]$, we compute further the following Lebesgue-Stieltjes integral as

$$\int_a^b f(x)\,dg(x) = \big(g(b)-g(a)\big)\int_a^b f(x)\,d\frac{g(x)-g(a)}{g(b)-g(a)} \in \big(g(b)-g(a)\big)[\min_{[a,b]} f, \max_{[a,b]} f].$$

Thus there exists $\xi \in [a,b]$ such that $\int_a^b f(x)\,dg(x) = f(\xi)\big(g(b)-g(a)\big)$, which is the second mean value theorem for Stieltjes integrals. The same result holds for monotonically decreasing function g.

Now let $F(x) = \int_a^x f(t)\,\mathrm{d}t$, then we use integration by parts and the second mean value theorem above to derive

$$\begin{aligned} \int_a^b f(x)g(x)\,\mathrm{d}x \,&= \int_a^b F'(x)g(x)\,\mathrm{d}x \,= [F(x)g(x)]_a^b - \int_a^b F(x)\,dg(x) \\ &= F(b)g(b) - F(\xi)\big(g(b)-g(a)\big) = g(b)\int_\xi^b f(x)\,\mathrm{d}x \,+ g(a)\int_a^\xi f(x)\,\mathrm{d}x\,. \end{aligned}$$

$$\leqslant \varepsilon\Big(\Big|\int_{(N+\frac{1}{2})\xi}^{(N+\frac{1}{2})\delta} \frac{\sin t}{\pi t}\,\mathrm{d}t\,\Big| + C\Big) \leqslant \tilde{C}\varepsilon,$$

while the second integral tends to 0 as $N \to \infty$ by Riemann-Lebesgue lemma. Hence (1.22) holds and correspondingly (1.21) follows. In particular, if f is continuous and has a bounded variation in a neighborhood of x, then

$$\lim_{N\to\infty} S_N f(x) = f(x).$$

The classical Dirichlet's pointwise convergence criterion (1829)[24]:

- f is bounded and piecewise continuous;
- f has a finite number of maxima and minima,

is a special case of Jordan's criterion.

1.3.4 Fejér's kernel

The pointwise convergece theory for Fourier series of continuous functions was "rescued" by L. Fejér in 1900 [25]. Instead of the natural partial sum of Fourier series where Dirichlet kernel D_N is involved, L. Fejér made use of the arithmetic means of partial sums (Cesàro's summation)

$$\sigma_N = \frac{1}{N+1}\sum_{k=0}^{N} S_k,$$

and the associated Fejér kernel

$$\begin{aligned} F_N(y) &= \frac{1}{N+1}\sum_{k=0}^{N} D_k(y) \\ &= \frac{1}{N+1}\frac{1}{2\pi}\frac{\sin(\frac{1}{2}y) + \sin(\frac{3}{2}y) + \sin(\frac{5}{2}y) + \cdots + \sin((N+\frac{1}{2})yss)}{\sin(\frac{1}{2}y)} \\ &= \frac{1}{N+1}\frac{1}{2\pi}\frac{\cos 0 - \cos((N+1)y)}{2(\sin(\frac{1}{2}y))^2} \\ &= \frac{1}{N+1}\frac{1}{2\pi}\Big(\frac{\sin(\frac{1}{2}(N+1)y)}{\sin(\frac{1}{2}y)}\Big)^2 \geqslant 0. \end{aligned}$$

[24] Peter Gustav Lejeune Dirichlet, Sur la convergence des séries trigonométriques qui servent à représenter une fonction arbitraire entre des limites données, *Journal für die reine und angewandte Mathematik* (Crelle's journal), 1829.

[25] Lipót Fejér, Sur les fonctions bornées et intégrables, *Comptes rendus hebdomadaires des séances de l'Académie des sciences*, 1900.

Note that unlike the sign-changing Dirichlet kernel D_N, $F_N \geqslant 0$ is nonnegative. F_N satisfies also (1.16):

$$\int_{-\pi}^{\pi} F_N(y)\,\mathrm{d}y \,= 1 \text{ and } F_N(y) \leqslant \frac{1}{N+1}\frac{1}{2\pi}\frac{1}{(\sin(\frac{1}{2}\delta))^2} \text{ if } \delta \leqslant |y| \leqslant \pi,$$

and is an approximate identity. Then we have immediately that for any continuous function $f \in C([-\pi,\pi])$,

$$\lim_{N\to\infty} \sup{}_{x\in[-\pi,\pi]} |(\sigma_N f)(x) - f(x)| = 0.$$

Indeed,

$$\begin{aligned}|(\sigma_N f)(x) - f(x)| &= \Big|\int_{-\pi}^{\pi} \big(f(x-t) - f(x)\big) F_N(t)\,\mathrm{d}t\Big| \\ &\leqslant \int_{|t|<\delta} |f(x-t) - f(x)| F_N(t)\,\mathrm{d}t \,+ 2\|f\|_\infty \int_{\delta\leqslant|t|\leqslant\pi} F_N(t)\,\mathrm{d}t\,,\end{aligned}$$

where δ can be chosen such that the first integral be arbitrarily small, while for any fixed δ, the second integral tends to 0 as N tends to ∞.
The above argument can be used to show the $L^p(-\pi,\pi)$ convergence of f by $\sigma_N f$ for $p \in [1,\infty)$, as long as one notes $\lim_{t\to 0} \|f(x-t) - f(x)\|_{L^p_x} = 0$ for $p \in [1,\infty)$. This implies the following facts:

- The trigonometric polynomials are dense in $L^p(-\pi,\pi)$, $p \in [1,\infty)$.
- Weierstrass approximation theorem[26]: Every continuous function on the torus is a uniform limit of trigonometric polynomials.
- If f is integrable with $c_n = 0$ for all $n \in \mathbb{Z}$, then f is identically zero (a.e.).

[Sep 16, 2026]
[Sep 22, 2026]

1.4 Convergence in norm of Fourier series

We have already achieved the satisfactory convergence result of Fourier series in the sense of $L^2(-\pi,\pi)$ by the spectral theorem:

$$\lim_{N\to\infty} \|S_N f - f\|_{L^2(-\pi,\pi)} = 0, \quad \forall f \in L^2(-\pi,\pi). \tag{1.23}$$

[26]Karl Theodor Wilhelm Weierstrass, *Über die analytische Darstellbarkeit sogenannter willkürlicher Functionen einer reellen Veränderlichen*, Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften zu Berlin (1885), 633–639, 789–805.

where the partial sum $S_N f$ is defined in (1.14). Furthermore, the mapping $f \mapsto (c_n)$ is an isometry from $L^2(-\pi, \pi)$ to $\ell^2(\mathbb{Z})$ by Parseval's identity[27] ⚠,

$$\|f\|^2_{L^2(-\pi,\pi)} = \sum_{n=-\infty}^{\infty} |c_n|^2. \tag{1.24}$$

We may also ask whether the convergence result (1.23) holds in the sense of $L^p(-\pi, \pi)$.

- Case $p = \infty$.

  Du Bois-Reymond's counterexample in Subsection 1.3.2 gives the negative answer for the case $p = \infty$.

- If $p \in [1, \infty)$, we have the following equivalence relation for $p \in [1, \infty)$:

$$\|S_N f - f\|_{L^p(-\pi,\pi)} \to 0 \text{ as } N \to \infty, \quad \forall f \in L^p(-\pi, \pi) \tag{1.25}$$
$$\Leftrightarrow \exists C = C_p \text{ such that uniformly in } N, \quad \|S_N\|_{L^p(-\pi,\pi)\to L^p(-\pi,\pi)} \leqslant C. \tag{1.26}$$

  The necessity of (1.26) follows from the uniform boundedness principle. The other direction follows from the following facts:

  - Density property: $C_0^\infty(-\pi, \pi)$ is dense in $L^p(-\pi, \pi)$;
  - Uniform convergence and hence convergence in L^p-norm for a dense subset $C_0^\infty(-\pi, \pi)$: For any $\varphi \in C_0^\infty(-\pi, \pi)$, $\|S_N\varphi - \varphi\|_{L^p(-\pi,\pi)} \leqslant (2\pi)^{\frac{1}{p}} \|S_N\varphi - \varphi\|_{L^\infty(-\pi,\pi)} \to 0$ as $N \to \infty$,

  and the decomposition $S_N f - f = S_N(f - \varphi) + (S_N\varphi - \varphi) + (\varphi - f)$. Therefore

  - Case $p = 1$.
    Similar as the analysis for the norm of T_N in (1.20)[28], we obtain

$$\|S_N\|_{L^1\to L^1} = \|D_N\|_{L^1(-\pi,\pi)} = O(\ln N) \text{ as } N \to \infty,$$

    and thus the convergence statement (1.25) fails for $p = 1$[29].

[27] Marc-Antoine Parseval des Chênes, *Mémoire sur les séries et sur l'intégration complète d'une équation aux différences partielles linéaires du second ordre à coefficients constants*, Mémoires de l'Académie des Sciences de l'Institut de France 2, 593–611, 1806.

[28] This is due to Young's inequality: $\|S_N f\|_{L^1} \leqslant \|D_N\|_{L^1}\|f\|_{L^1}$ and the result (1.20): $\|S_N\|_{L^1\to L^1} = \|S_N^*\|_{L^\infty\to L^\infty} \geqslant \|D_N\|_{L^1}$.

[29] Andrey Nikolaevich Kolmogorov, Une série de Fourier–Lebesgue divergente presque partout. *Fundamenta Mathematicae* 4, 324–328, 1923.

– Case $p \in (1, \infty)$.

We claim that the convergence result (1.25) for $p \in (1, \infty)$ follows from the L^p-boundedness of the conjugate function discussed in Subsection 1.4.1 below, that is,

$$\|S_N - \mathrm{Id}\,\|_{L^p \to L^p} \overset{N\to\infty}{\to} 0 \Leftrightarrow \|S_N\|_{L^p\to L^p} \leqslant C_p \Leftarrow \|\mathcal{H}\|_{L^p\to L^p} \leqslant \tilde{C}_p.$$

1.4.1 Conjugate function and Riesz projections

In the functional setting of $L^2(-\pi, \pi)$, we have shown (1.13):

$$f(x) = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} \widehat{f}(n) e^{inx}, \quad \widehat{f}(n) := c_n = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(x) e^{-inx}\, \mathrm{d}x\,. \quad (1.27)$$

We define the conjugate function of a function f as[30][31]

$$\mathcal{H}(f)(x) = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} (\frac{1}{i}\,\mathrm{sgn}\,(n)) \widehat{f}(n) e^{inx}, \quad (1.28)$$

where $\mathrm{sgn}\,(0) = 0$, as well as Riesz projections $P_\pm$ by [32]

$$P_+(f)(x) = \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \widehat{f}(n) e^{inx}, \quad (1.29)$$

$$P_-(f)(x) = -\frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{-1} \widehat{f}(n) e^{inx}. \quad (1.30)$$

[30]Historically it was the concept of harmonic conjugate (see (1.31)-(1.36) below) appeared first, and the pair (u, v) satisfies the Cauchy-Riemann equations. The conjugate function $\mathcal{H}(f)$ happens to be the limit of the harmonic conjugate v on the unit circle (see (1.38) below).

Augustin-Louis Cauchy, Mémoire sur les intégrales définies prises entre des limites imaginaires, de Bure Frères, Paris, 1825.

Bernhard Riemann, Grundlagen für eine allgemeine Theorie der Functionen einer veränderlichen complexen Grösse, Inauguraldissertation, Göttingen, 1851; reprinted in Gesammelte mathematische Werke und wissenschaftlicher Nachlass, B. G. Teubner, Leipzig, 3–48, 1876.

[31]We will discuss the counter part, namely the Hilbert transform, on the whole line in Subsection 4.1.3 later.

[32]This is after Marcel Riesz (1924, 1927), to recognize his contribution to the L^p-boundedenss of the conjugate function for $p \in (1, \infty)$.

The Riesz projection $P = \frac{1}{2\pi i} \oint_\Gamma (\lambda I - T)^{-1} d\lambda$, which is after his brother Frigyes Riesz, is related to the spectral projection which we discussed earlier in Section 1.2.

Hardy space (1915) L^2_+ consists of L^2 functions with Fourier transform supported on the domain $\{n \geqslant 0\}$, that is $L^2_+ = \{P_+ f \,|\, f \in L^2\}$. It identifies holomorphic functions F on the unit disc such that $\sup_{r<1} \int_{-\pi}^{\pi} |F(re^{ix})|^2\, \mathrm{d}x < \infty$.

Then the following relations hold

$$f = P_+(f) - P_-(f), \quad P_+(f) + P_-(f) = i\mathcal{H}(f) + \frac{1}{\sqrt{2\pi}}\widehat{f}(0),$$

or equivalently,

$$P_\pm(f) = \frac{1}{2}\Big(\pm f + i\mathcal{H}(f) + \frac{1}{\sqrt{2\pi}}\widehat{f}(0)\Big).$$

We now reformulate the partial sum S_N in terms of Riesz operator P_+ and multiplication operators e^{iNx}, e^{i2Nx}. We first rewrite

$$\begin{aligned} S_N f &= \frac{1}{\sqrt{2\pi}} \sum_{n=-N}^{N} \widehat{f}(n) e^{inx} \\ &= \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{2N} \widehat{f}(n-N) e^{i(n-N)x} \\ &= e^{-iNx} \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{2N} \widehat{f(\cdot)e^{iN\cdot}}(n) e^{inx}, \end{aligned}$$

where we compute for any $g \in L^2(-\pi, \pi)$ that

$$\begin{aligned} \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{2N} \widehat{g}(n) e^{inx} &= \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \widehat{g}(n) e^{inx} - \frac{1}{\sqrt{2\pi}} \sum_{n=2N+1}^{\infty} \widehat{g}(n) e^{inx} \\ &= P_+(g) - e^{i(2N+1)x} \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \widehat{g(\cdot)e^{-i(2N+1)\cdot}}(n) e^{inx} \\ &= P_+(g) - e^{i(2N+1)x} P_+(g e^{-i(2N+1)\cdot}). \end{aligned}$$

To conclude, we can represent the partial sum operator S_N as

$$S_N f = e^{-iNx} P_+(f e^{iN\cdot}) - e^{i(N+1)x} P_+(f e^{-i(N+1)\cdot}).$$

Since the multiplication operator e^{iax}, $a \in \mathbb{R}$ is a bounded operator on L^p-space with operator norm 1, we derive for all $N \in \mathbb{N}$,

$$\|S_N\|_{L^p(-\pi,\pi)\to L^p(-\pi,\pi)} \leqslant 2\|P_+\|_{L^p(-\pi,\pi)\to L^p(-\pi,\pi)},$$

which is bounded by the norm $\|\mathcal{H}\|_{L^p(-\pi,\pi)\to L^p(-\pi,\pi)}$, up to some constants. The fact[33] that $\mathcal{H}$ is a bounded operator in $L^p(-\pi, \pi)$, $p \in (1, \infty)$ implies the uniform L^p-boundedness of S_N in (1.26), and hence the convergence result (1.25) holds for $p \in (1, \infty)$.

[33]We don't give the proof here, which can be found e.g. in [2, Subsection 4.1.2].

Historically, the $L^p(-\pi,\pi), p\in(1,\infty)$-norm convergence of Fourier series was established by M. Riesz (1927) [34], who proved the L^p boundedness of the conjugate function for $p\in(1,\infty)$. A modern proof rests on the Marcinkiewicz interpolation theorem[35] interpolating the L^2 isometry (1.24) against Kolmogorov's weak $(1,1)$ bound[36] yields the strong (p,p) inequality for $p\in(1,2)$, while the range $p\in(2,\infty)$ follows by duality, the adjoint of the conjugation operator being its own negative.
The real-variable form of the argument is due to Calderón and Zygmund[37], whose decomposition replaced the complex-analytic proofs and supplied the weak $(1,1)$ estimate for general singular integrals in $\mathbb{R}^n$; in this formulation the decisive object is the maximal function[38], $\sup_{r<1}|P_r f|$ for the Poisson kernel and $\sup_{t>0}|Q_t f|$ for the conjugate function, whose weak $(1,1)$ and strong (p,p) bounds govern pointwise and norm behaviour alike, and whose partial-sum analogue $\sup_N|S_N f|$ is precisely what later carried the problem from norm convergence to almost-everywhere convergence.

1.4.2 Harmonic extension and Poisson kernel

We turn to view Fourier series from another perspective, by considering the following function defined in the unit disk on the complex plane:

$$u(z)=\frac{1}{\sqrt{2\pi}}\Big(\sum_{n=0}^{\infty}\widehat{f}(n)\,z^n+\sum_{n=-\infty}^{-1}\widehat{f}(n)\,\bar{z}^{|n|}\Big),\quad |z|<1, \tag{1.31}$$

that is,

$$u(re^{ix})=\frac{1}{\sqrt{2\pi}}\sum_{n=-\infty}^{\infty}\widehat{f}(n)r^{|n|}e^{inx},\quad r\in[0,1),\quad x\in[-\pi,\pi].$$

Thus the Fourier series can be viewed as the formal limit of the harmonic function u on the unit circle,

$$\frac{1}{\sqrt{2\pi}}\sum_{n=-\infty}^{\infty}\widehat{f}(n)\,e^{inx}=\lim_{r\to1_-}u(re^{ix}),\qquad x\in[-\pi,\pi]. \tag{1.32}$$

[34] Marcel Riesz, Sur les fonctions conjuguées, *Math. Z.* 27, 218–244, 1927 (announced in Les fonctions conjuguées et les séries de Fourier, *Comptes Rendus de l'Académie des Sciences de Paris* 178, 1464–1467, 1924).

[35] Józef Marcinkiewicz, Sur l'interpolation d'opérateurs, *Comptes Rendus de l'Académie des Sciences de Paris* 208, 1272–1273, 1939.

[36] Andrey Nikolaevich Kolmogorov, Sur les fonctions harmoniques conjuguées et les séries de Fourier, *Fund. Math.* 7, 23–28, 1925.

[37] Alberto Pedro Calderón and Antoni Zygmund, On the existence of certain singular integrals, *Acta Math.* 88, 85–139, 1952.

[38] Godfrey Harold Hardy and John Edensor Littlewood, A maximal theorem with function-theoretic applications, *Acta Math.* 54, 81–116, 1930.

When f is real-valued, $\widehat{f}(-n) = \overline{\widehat{f}(n)}$, u is a real-valued harmonic function. We now analyze the limit on the right hand side. Rewrite, for $r < 1$,

$$u(re^{ix}) = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} \widehat{f}(n) r^{|n|} e^{inx} = \int_{-\pi}^{\pi} f(x) P_r(x-t)\ \mathrm{d}t\,,$$

where $P_r(t)$ denotes the Poisson kernel[39] [40][41]

$$P_r(t) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} r^{|n|} e^{int}.$$

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It can be computed explicitly by use of complex variables for $r \in [0,1)$: Observe that for $\zeta = re^{it}$ with $|\zeta| < 1$,

$$\begin{aligned} P_r(t) &= \frac{1}{2\pi} \mathrm{Re}\left[1 + 2\sum_{n=1}^{\infty} \zeta^n\right] = \frac{1}{2\pi} \mathrm{Re}\left[\frac{1+\zeta}{1-\zeta}\right] \\ &= \frac{1}{2\pi} \mathrm{Re}\left[\frac{(1+\zeta)(1-\bar{\zeta})}{|1-\zeta|^2}\right] = \frac{1}{2\pi} \frac{1-|\zeta|^2}{|1-re^{it}|^2} \\ &= \frac{1}{2\pi} \frac{1-r^2}{1-2r\cos t + r^2} > 0. \end{aligned} \tag{1.33}$$

Note that similar as the Fejér kernel, $\{P_r\}_{r\in(0,1)}$ is an approximate identity, satisfying

$$P_r(t) \geqslant 0, \quad \int_{-\pi}^{\pi} P_r(t)\,\mathrm{d}t\ = 1, \quad \lim_{r\to 1_-} \int_{\delta\leqslant|t|\leqslant\pi} P_r(t)\,\mathrm{d}t\ = 0 \text{ if } \delta > 0. \tag{1.34}$$

Therefore the argument in Subsection 1.3.4 yields the limit on the right hand side of (1.32) in the following sense:

[39] Siméon Denis Poisson, Mémoire sur l'intégration de quelques équations linéaires aux différences partielles, et particulièrement de l'équation générale du mouvement des fluides élastiques, *Mémoires de l'Académie Royale des Sciences de l'Institut de France* **3** (1820/1823), 121–176.

[40] Karl Hermann Amandus Schwarz, Über die Integration der partiellen Differentialgleichung $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0$ unter vorgeschriebenen Grenz- und Unstetigkeitsbedingungen,*Monatsberichte der Königlich Preussischen Akademie der Wissenschaften zu Berlin*, 767–795, 1870.

[41] Formally, "$\lim_{r\to 1_-} P_r = \lim_{N\to\infty} D_N$".

- Case $p \in [1, \infty)$.

  If $f \in L^p(-\pi, \pi)$, then

$$\lim_{r \to 1_-} \|u(re^{ix}) - f(x)\|_{L^p} = \lim_{r \to 1_-} \|P_r * f - f\|_{L^p} = 0;$$

- Case $p = \infty$.

  If $f \in C([-\pi, \pi])$ is continuous, then

$$\lim_{r \to 1_-} \|u(re^{ix}) - f(x)\|_{L^\infty} = \lim_{r \to 1_-} \|P_r * f - f\|_{L^\infty} = 0.$$

Note that if f is real-valued, then the function u defined in terms of f in (1.31) is harmonic function inside the unit disk. The above convergence result implies that so-defined function $u(re^{ix}) = (P_r * f)(x)$ is the (unique) solution of the Dirichlet problem for the Laplace equation on the unit disk:

$$\begin{cases} \Delta u = 0 & \text{if } |z| < 1, \\ u = f & \text{if } |z| = 1, \end{cases} \tag{1.35}$$

where the boundary condition is satisfied in the above sense. The function u is thus called the harmonic extension of f inside the unit disk.

1.4.3 Harmonic conjugate and conjugate Poisson kernel

We continue the study of the harmonic function $u(z) = u(re^{ix})$ defined by (1.31) inside the unit disk. Let

$$v(z) = \frac{1}{\sqrt{2\pi}} \Big(\sum_{n=1}^{\infty} \frac{1}{i} \widehat{f}(n)\, z^n - \sum_{n=-\infty}^{-1} \frac{1}{i} \widehat{f}(n)\, \bar{z}^{|n|} \Big), \quad |z| < 1, \tag{1.36}$$

that is,

$$v(re^{ix}) = \frac{1}{\sqrt{2\pi}} \sum_{n \in \mathbb{Z}} \big(\frac{1}{i} \operatorname{sgn}(n) \big) \widehat{f}(n) r^{|n|} e^{inx}, \quad r \in [0, 1), \quad x \in [-\pi, \pi].$$

Observe that

- If $\{\widehat{f}(n)\}$ is a bounded sequence, then v is the harmonic conjugate of u, in the sense that

$$G(z) = u(z) + iv(z) = \frac{1}{\sqrt{2\pi}} \Big(\widehat{f}(0) + 2 \sum_{n=1}^{\infty} \widehat{f}(n)\, z^n \Big), \tag{1.37}$$

  is holomorphic in the unit disk.

- Formally, the conjugate function defined in (1.28) can be viewed as the limit of $v(z)$ on the unit circle:

$$\mathcal{H}(f)(x) = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} (\frac{1}{i}\mathrm{sgn}\,(n))\widehat{f}(n)e^{inx} = \lim_{r\to 1_-} v(re^{ix}). \tag{1.38}$$

Similarly as for u, we rewrite as a convolution

$$v(re^{ix}) = \int_{-\pi}^{\pi} f(t)Q_r(x-t)\,\mathrm{d}t \;= (Q_r * f)(x)$$

with the conjugate Poisson kernel

$$Q_r(t) = \frac{1}{2\pi}\sum_{n\in\mathbb{Z}}\big(\frac{1}{i}\,\mathrm{sgn}(n)\big)r^{|n|}e^{int}, \quad r\in[0,1), \quad t\in[-\pi,\pi].$$

It is real-valued and we compute by use of complex variable $\zeta = re^{it}$ with $|\zeta| = r < 1$, similarly as for Poisson kernel:

$$\begin{aligned} Q_r(t) &= \frac{1}{2\pi}\mathrm{Im}\Big[\sum_{n=1}^{\infty}\zeta^n - \sum_{m=1}^{\infty}\bar{\zeta}^m\Big] = \frac{1}{2\pi}\mathrm{Im}\Big[1+2\sum_{n=1}^{\infty}\zeta^n\Big] \\ &= \frac{1}{2\pi}\mathrm{Im}\Big[\frac{1+\zeta}{1-\zeta}\Big] = \frac{1}{2\pi}\mathrm{Im}\Big[\frac{(1+\zeta)(1-\bar{\zeta})}{|1-\zeta|^2}\Big] \\ &= \frac{1}{2\pi}\frac{2r\sin t}{1-2r\cos t+r^2}. \end{aligned}$$

Note that

- $P_r(t)+iQ_r(t)$ is holomorphic function in ζ in the unit disk:

  $$P_r(t)+iQ_r(t) = \frac{1}{2\pi}\frac{1+\zeta}{1-\zeta} = \frac{1}{2\pi}\big(-1+\frac{2}{1-\zeta}\big), \text{ with } \zeta = re^{it},\ r\in[0,1),$$

  such that the holomorphic function $G(z)$ can be represented by it as

  $$\begin{aligned} G(z) = u(re^{ix}) + iv(re^{ix}) &= \big((P_r+iQ_r)*f\big)(x) \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\Big(-1+\frac{2}{1-re^{i(x-t)}}\Big)f(t)\,\mathrm{d}t \\ &\overset{w=e^{it}}{=} -\frac{1}{\sqrt{2\pi}}\widehat{f}(0) + 2\cdot\frac{1}{2\pi i}\int_{|w|=1}\frac{1}{w-z}f(t)dw. \end{aligned}$$

  Here the integral is indeed the Cauchy integral of the function f on the unit circle:

$$F(z) = \frac{1}{2\pi i}\int_{|w|=1}\frac{f(t)}{w-z}dw \overset{w=e^{it}}{=} \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{f(t)e^{it}}{e^{it}-z}\,\mathrm{d}t\,. \tag{1.39}$$

- Unlike P_r, the kernel Q_r is not an approximate identity:
  - It has zero mean $\int_{-\pi}^{\pi} Q_r(t)\,\mathrm{d}t = 0$;
  - Its L^1-norm diverges as r tends to 1_-:

$$\begin{aligned}\|Q_r\|_{L^1(-\pi,\pi)} &= \frac{2}{\pi}\int_0^{\pi} \frac{r\sin t}{1-2r\cos t+r^2}\,\mathrm{d}t \overset{y=2r\cos t}{=} \frac{1}{\pi}\int_{-2r}^{2r}\frac{1}{1+r^2-y}\,\mathrm{d}y\\ &= \frac{1}{\pi}\ln(1+r^2-y)|_{2r}^{-2r} = \frac{1}{\pi}\ln\frac{(1+r)^2}{(1-r)^2}\\ &\sim \frac{2}{\pi}\log\frac{1}{1-r} \to \infty \text{ as } r\to 1_-.\end{aligned}$$

  Hence the argument used for P_r breaks down, and this failure is exactly where the difficulty of the convergence problem lives. Indeed, the convergence (1.38): $\mathcal{H}(f)(x) = \lim_{r\to 1_-} Q_r * f(x)$ holds in L^p and almost everywhere for $f\in L^p$, $1\leqslant p<\infty$, and the theorem of M. Riesz states that the map $f\mapsto \mathcal{H}(f)$ is bounded on L^p for $p\in(1,\infty)$.

1.4.4 Other related topics

Boundary traces of the Cauchy integral from inside and outside Formally, the Fourier coefficients of $f, \mathcal{H}(f)$ and that of $u(re^{ix}), v(re^{ix})$ are related as

$$\forall r\in(0,1), \widehat{u(re^{i\cdot})}(n) = r^{|n|}\hat{f}(n), \text{ which converges to } \hat{f}(n) \text{ as } r\to 1_-,$$
$$\forall r\in(0,1), \widehat{v(re^{i\cdot})}(n) = (\frac{1}{i}\mathrm{sgn}\,(n))r^{|n|}\hat{f}(n), \text{ which converges to } \hat{\mathcal{H}(\{})(n) \text{ as } r\to 1_-.$$

Correspondingly, the relations between the operators $\mathcal{H}, \mathrm{Id}$ and the functions P_r, Q_r can be seen more clearly in terms of their Fourier transforms[42]:

$$\sqrt{2\pi}\widehat{P_r}(n) = r^{|n|}, \quad \lim_{r\to 1_-}\sqrt{2\pi}\widehat{P_r}(n) = 1(n) = \widehat{\mathrm{Id}},$$
$$\sqrt{2\pi}\widehat{Q_r}(n) = \frac{1}{i}\mathrm{sgn}\,(n)r^{|n|}, \quad \lim_{r\to 1_-}\sqrt{2\pi}\widehat{Q_r}(n) = \frac{1}{i}\mathrm{sgn}\,(n) = \widehat{\mathcal{H}}(n).$$

We remark that the Riesz projections $P_\pm$:

$$P_\pm = \frac{1}{2}\Big(\pm\mathrm{Id} + i\mathcal{H} + \frac{1}{2\pi}\int_{-\pi}^{\pi}\Big)$$

[42]Notice $\widehat{u}(n) = \sqrt{2\pi}\widehat{P_r}(n)\cdot\hat{f}(n)$ and $\widehat{v}(n) = \sqrt{2\pi}\widehat{Q_r}(n)\cdot\hat{f}(n)$.

with

$$\widehat{P_+}(n) = 1_{\{0,1,\cdots\}}(n), \quad \widehat{P_-}(n) = -1_{\{-1,-2,-3,\cdots\}}(n),$$

correspond to the traces $\lim_{r\to 1_\mp}$ of the Cauchy integral of f: $F(z)$ given in (1.39). Indeed, for $|z|<1$, expanding $\dfrac{1}{w-z} = \sum_{n=0}^\infty \dfrac{z^n}{w^{n+1}} = \sum_{n=0}^\infty e^{-i(n+1)t} z^n$:

$$F_+(z) := F(z) = \sum_{n=0}^\infty \Big(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t) e^{-int}\,\mathrm{d}t\Big) z^n = \sum_{n=0}^\infty \frac{1}{\sqrt{2\pi}} \widehat{f}(n)\, z^n, \quad |z|<1.$$

For $|z|>1$, expanding in $\dfrac{1}{w-z} = -\sum_{n=1}^\infty \dfrac{w^{n-1}}{z^n} = -\sum_{n=1}^\infty e^{i(n-1)t} z^{-n}$:

$$\begin{aligned} F_-(z) := F(z) &= -\sum_{n=1}^\infty \Big(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t) e^{int}\,\mathrm{d}t\Big) z^{-n} = -\sum_{n=1}^\infty \frac{1}{\sqrt{2\pi}} \widehat{f}(-n)\, z^{-n} \\ &= -\frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{-1} \widehat{f}(n)\, z^n, \quad |z|>1. \end{aligned}$$

Thus by the definitions (1.29) and (1.30), $P_\pm(f)(x) = \lim_{r\to 1_\mp} F_\pm(re^{ix})$. Hence the function f can be recovered from its Cauchy integral F by taking the jump on the boundary: $f(x) = \lim_{r\to 1_-} F(re^{ix}) - \lim_{r\to 1_+} F(re^{ix})$.

Almost everywhere convergence The L^p-convergence (1.25) does not imply the almost everywhere convergence[43]. It takes a long time for mathematicians to achieve the almost everywhere convergence result of Fourier series for the function $f \in L^p(-\pi,\pi)$, $p \in (1,\infty]$.

- Case $p=1$ is excluded, due to the existence of L^1-functions whose Fourier series diverge everywhere, which is constructed by Andrey Nikolaevich Kolmogorov in 1926[44].
- Case $p=2$ is included, thanks to Lennart Carleson's work in 1966[45].

  Since $L^p(-\pi,\pi) \subset L^2(-\pi,\pi)$ for all $p \in [2,\infty]$, the almost everywhere convergence holds for the cases $p \in [2,\infty]$.

[43] It holds only up to a subsequence.

[44] Andrey Nikolaevich Kolmogorov, Une série de Fourier–Lebesgue divergente partout. *Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences (Paris)* 183, 1327–1329, 1926.

Andrey Nikolaevich Kolmogorov, Une série de Fourier – Lebesgue divergente presque partout. *Fundamenta Mathematicae* 4, 324–328, 1923.

[45] Lennart Carleson, On convergence and growth of partial sums of Fourier series. *Acta Mathematica* 116, 135–157, 1966.

- Case $p \in (1,2)$ is included, thanks to R.A. Hunt's result in 1967[46].

Higher dimensional case We can of course the higher dimensional Fourier series. We denote $L^p_{\text{per}}(\mathbb{R}^d)$, $p \in [1,\infty]$ to be the space of measurable functions $f : \mathbb{R}^d \mapsto \mathbb{C}$ which are periodic such that $f \in L^p([-\pi,\pi]^d)$. We identify the two $L^p_{\text{per}}(\mathbb{R}^d)$ functions if they are the same almost everywhere. We define the k, $k = (k_1, \cdots, k_d)$-th Fourier coefficient of a function $f \in L^1_{\text{per}}(\mathbb{R}^d)$ by

$$\hat{f}(k) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{[-\pi,\pi]^d} f(x) e^{-ik\cdot x} \,\mathrm{d}x\,,$$

and we call the following sequence $(S_N f(x))_N$ the Fourier series of f:

$$S_N f(x) = \frac{1}{(2\pi)^{\frac{d}{2}}} \sum_{|k|\leqslant N} \hat{f}(k) e^{ik\cdot x}, \quad N \in \mathbb{N}\,.$$

From $[-\pi,\pi]$ to $\mathbb{R}$ Intuitively, we go from sums to integrals (by considering T-periodic functions with $T \to \infty$), to realize the limit of the Fourier series as the inverse Fourier transform of its Fourier transform.
Formally, for $d = 1$, let T be the period, and $\omega = \frac{2\pi}{T}$ be the frequence, then the Fourier coefficient of a function $f \in L^1([-\frac{T}{2},\frac{T}{2}])$ is defined as

$$\hat{f}(k) = \frac{1}{\sqrt{T}} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(x) e^{-ik\omega x} \,\mathrm{d}x \;= \frac{\sqrt{\omega}}{\sqrt{2\pi}} \int_{-\frac{\pi}{\omega}}^{\frac{\pi}{\omega}} f(x) e^{-i(k\omega)x} \,\mathrm{d}x\,,$$

and the representation of f in terms of Fourier series reads as

$$\begin{aligned} f(x) &= \frac{\sqrt{\omega}}{\sqrt{2\pi}} \sum_k \hat{f}(k) e^{i(k\omega)x} \\ &= \frac{1}{\sqrt{2\pi}} \sum_k \Big(\frac{1}{\sqrt{2\pi}} \int_{-\frac{\pi}{\omega}}^{\frac{\pi}{\omega}} f(y) e^{-i(k\omega)y} \,\mathrm{d}y \Big) e^{i(k\omega)x} \omega. \end{aligned}$$

Let $\omega \to 0_+$, we formally obtain from the definition of integral that

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \Big(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y) e^{-iy\xi} \,\mathrm{d}y \Big) e^{ix\xi} \,\mathrm{d}\xi\,.$$

[46]Richard A. Hunt, On the convergence of Fourier series. In Orthogonal Expansions and their Continuous Analogues (Proc. Conf., Edwardsville, Ill., 1967), Southern Illinois University Press, 235–255.

We then derive the continuous version of Fourier coefficients and Fourier series

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ix\xi} \, \mathrm{d}x \,, \quad \xi \in \mathbb{R},$$
$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f}(\xi) e^{ix\xi} \, \mathrm{d}\xi \,, \quad x \in \mathbb{R} \,.$$

This is the Fourier transform on $\mathbb{R}$, and we will discuss it later in Chapter 3 in more detail. To this end, we turn to the introduction of related functional spaces.

[Sep 23, 2026]
[Sep 29, 2026]

2 Function spaces

In this section, we restrict ourselves to the functional framework of (generalized) functions defined on the whole Euclidean space $\mathbb{R}^d$[47] and taking values in $\mathbb{K}$ with $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. The notation K denotes some compact set in $\mathbb{R}^d$. The goal is to generalize the concept of functions by duality approach.

2.1 Continuous function framework

We introduce some classical continuous function spaces, with particular attention to their dual spaces.

2.1.1 Continuous functions

The continuous bounded function space

$$C_b(\mathbb{R}^d) = \{f : \mathbb{R}^d \to \mathbb{K} \text{ continuous and bounded}\},$$

is Banach space, equipped with the norm

$$\|f\|_{C_b(\mathbb{R}^d)} := \|f\|_\infty = \sup_{x \in \mathbb{R}^d} |f(x)|.$$

The space

$$C_c(\mathbb{R}^d) = \{f : \mathbb{R}^d \to \mathbb{K} \text{ continuous and with compact support}\},$$

[47] It is straightforward to generalize the definitions from $\mathbb{R}^d$ to any open subset $U \subset \mathbb{R}^d$.

is a vector subspace but not a closed subset of $C_b(\mathbb{R}^d)$. The closure of it is denoted by

$$C_0(\mathbb{R}^d) = \overline{C_c(\mathbb{R}^d)}^{\|\cdot\|_\infty},$$

which consist of continuous functions f with limit 0 at ∞, i.e. for all $\varepsilon > 0$ there exists j so that f is at most of size ε outside the ball B_j, which is centered at the origin and of radius j.

We take another topology in $C_c(\mathbb{R}^d)$ that we call

$$f_n \to f \text{ in } C_c(\mathbb{R}^d)$$

if there exists a compact set $K \subset \mathbb{R}^d$ such that $\mathrm{Supp}\,(f_n) \subset K$ for all n and

$$\|f_n - f\|_{C_b(\mathbb{R}^d)} \to 0.$$

A map $\mathcal{L} : C_c(\mathbb{R}^d) \to \mathbb{K}$ is called continuous if $\mathcal{L}(f_n) \to \mathcal{L}(f)$ holds for all $f_n \to f$ in $C_c(\mathbb{R}^d)$. Thus a continuous linear map on $C_c(\mathbb{R}^d)$ is bounded on $C_b(K)$ for all compact set $K \subset \mathbb{R}^d$. The Riesz representation theorem[48] for $C_c(\mathbb{R}^d)$ states as: Any nonnegative continuous linear map $\mathcal{L} \in \big(C_c(\mathbb{R}^d)\big)'$ is represented uniquely by a Radon measure μ on $\mathbb{R}^d$, in the sense that[49]

$$\mathcal{L}(f) = \int_{\mathbb{R}^d} f d\mu, \quad f \in C_c(\mathbb{R}^d).$$

Given $k \in \mathbb{N}$, we define

$$C_b^k(\mathbb{R}^d) = \{f : \mathbb{R}^d \to \mathbb{K} \mid f \text{ is } k \text{ times continuously differentiable and all partial derivatives } \partial^\alpha f \text{ with } |\alpha| \leqslant k \text{ are bounded}\}.$$

It is a Banach space, equipped with the norm

$$\|f\|_{C_b^k(\mathbb{R}^d)} := \|f\|_{k,\infty} = \sup_{|\alpha|\leqslant k} \|\partial_x^\alpha f(x)\|_\infty.$$

Similarly we define

$$C_c^k(\mathbb{R}^d) = \{f \in C_b^k(\mathbb{R}^d) \text{ has compact support}\}.$$

[48] See Theorem 2.14 in the book by Walter Rudin, *Real and Complex Analysis*, 3rd ed., McGraw-Hill, New York, 1987.

The continuity assumption is omitted in the statement there and it follows from the positivity of the functional: For any compact set K, there exists a compactly supported continuous function φ_K such that for all $f \in C_c(K)$, $\mathcal{L}(f) \leqslant \mathcal{L}(|f|) \leqslant \mathcal{L}(\|f\|_\infty \varphi_K) = \mathcal{L}(\varphi_K)\|f\|_\infty$.

[49] Bourbaki's equivalent viewpoint: a Radon measure is a continuous linear functional on $C_c(X)$ (for the inductive-limit topology).

2.1.2 Test functions and distributions

We are motivated to generalize the definitions of "functions" by studying the continuous linear maps on the test function space $C_c^\infty(\mathbb{R}^d)$. The vector space $C_c^\infty(\mathbb{R}^d)$ is not a Banach space, and its topology is complicated, which we do not discuss in detail here. [50]

Test function space We denote by $\mathcal{D}(\mathbb{R}^d) = C_c^\infty(\mathbb{R}^d)$ the vector space of all test functions on $\mathbb{R}^d$, which are infinitely differentiable functions with compact support in $\mathbb{R}^d$. A typical example is the function

$$\eta(x) = \begin{cases} c_d e^{\frac{1}{|x|^2-1}} & \text{if } |x| < 1, \\ 0 & \text{if } |x| \geqslant 1, \end{cases} \tag{2.40}$$

where $c_d \in \mathbb{R}^+$ is a constant such that $\int_{\mathbb{R}^d} \eta(x)\,\mathrm{d}x = 1$. We can rescale it by $\varepsilon > 0$: $\eta_\varepsilon = \frac{1}{\varepsilon^d}\eta(\frac{\cdot}{\varepsilon})$, then $(\eta_\varepsilon)_\varepsilon$ is an approximation of identity.
We can not endow test function space with approximate norm such that it becomes a Banach space. Nevertheless, we call

$$\varphi_j \to \varphi \text{ in } C_c^\infty(\mathbb{R}^d) = \mathcal{D}(\mathbb{R}^d)$$

if there is a compact set $K \subset \mathbb{R}^d$ so that $\operatorname{Supp}\varphi_j \subset K$ for all j, and for all multiindices α

$$\partial^\alpha \varphi_j \to \partial^\alpha \varphi \qquad \text{in } C_b(\mathbb{R}^d).$$

Distribution space A distribution T on $\mathbb{R}^d$ is a continuous linear map from $C_c^\infty(\mathbb{R}^d)$ to $\mathbb{K}$, where by continuity we mean

$$T(\varphi_j) \to T(\varphi), \quad \text{if } \varphi_j \to \varphi \text{ in } C_c^\infty(\mathbb{R}^d),$$

or equivalently, for any compact set K in $\mathbb{R}^d$, there exists $k \in \mathbb{N}$ and $C > 0$ such that

$$|T(\varphi)| \leqslant C\|\varphi\|_{C_b^k(K)}, \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^d) \text{ with } \operatorname{Supp}(\varphi) \subset K.$$

[50] Jean Dieudonné and Laurent Schwartz, La dualité dans les espaces $(\mathcal{F})$ et $(\mathcal{LF})$, Annales de l'Institut Fourier 1, 61–101, 1949.
Nicolas Bourbaki, Espaces vectoriels topologiques, Hermann, Paris, 1953–1955.
Nicolas Bourbaki, Intégration, Hermann, Paris, 1952–1969.
John Horváth, Topological Vector Spaces and Distributions, Vol. I, Addison-Wesley, Reading, 1966.
张平、邵瑞杰，广义函数与函数空间导论，科学出版社，2025.

We denote the space of distributions by $\mathcal{D}'(\mathbb{R}^d)$, (i.e. the dual space of $\mathcal{D}(\mathbb{R}^d)$), which is a $\mathbb{K}$-vector space. We call $T_n \to T$ in $\mathcal{D}'(\mathbb{R}^d)$ if

$$T_n(\varphi) \to T(\varphi), \qquad \forall \varphi \in \mathcal{D}(\mathbb{R}^d).$$

If $T_n \to T$ in $\mathcal{D}'(\mathbb{R}^d)$, then for any compact set $K \subset \mathbb{R}^d$, there exists k and C such that

- $|T_n(\varphi)| \leqslant C\|\varphi\|_{C_b^k(K)}, \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^d)$ with $\mathrm{Supp}\,(\varphi) \subset K$,
- $\sup_{\varphi \in \mathcal{D}(\mathbb{R}^d) \text{ with } \mathrm{Supp}\,(\varphi) \subset K, \|\varphi\|_{C_b^k(K)} \leqslant 1} |T_n(\varphi) - T(\varphi)| \to 0.$

Note that there is a natural local approximation $(\phi_j)_j \subset \mathcal{D}(\mathbb{R}^d)$ of the constant function 1 satisfying:

$$\mathrm{Supp}\,(\phi_j) \subset B_{2^{j+1}}, \quad \phi_j|_{B_{2^{j-1}}} = 1 \text{ such that } \phi_j \to 1 \text{ in } \mathcal{D}'(\mathbb{R}^d), \tag{2.41}$$

e.g. $\phi_j = \eta * 1_{B_{2^j}}$, $j \in \mathbb{N}$ does this job.
For $T \in \mathcal{D}'(\mathbb{R}^d)$, we say that T vanishes near $x \in \mathbb{R}^d$ if there exists $r > 0$ so that $T(\varphi) = 0$ for all $\varphi \in \mathcal{D}(\mathbb{R}^d)$ with support in $B_r(x)$, and we define

$$\mathrm{Supp}\,(T) = \mathbb{R}^d \setminus \{x \in \mathbb{R}^d \mid T \text{ vanishes near } x\}.$$

Examples of distributions Any Radon measure μ on $\mathbb{R}^d$ defines a unique distribution $T_\mu \in \mathcal{D}'(\mathbb{R}^d)$:

$$\langle T_\mu, \varphi \rangle_{\mathcal{D}',\mathcal{D}} := T_\mu(\varphi) = \int_{\mathbb{R}^d} \varphi d\mu, \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^d), \tag{2.42}$$

such that for any compact set $K \subset \mathbb{R}^d$,

$$|T_\mu(\varphi)| \leqslant \mu(K)\|\varphi\|_{C_b}, \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^d) \text{ with } \mathrm{Supp}\,(\varphi) \subset K.$$

Recall here that the integral on the right hand side can be built as follows:

Step 1 For simple functions $s = \sum_{j=1}^N c_j 1_{A_j}$, with $c_j \in \mathbb{K}$ and $A_j \subset \mathbb{R}^d$ pairwise disajoint Borel sets, we define

$$\int_{\mathbb{R}^d} s \, d\mu = \sum_{j=1}^N c_j \mu(A_j);$$

Step 2 For nonnegative $f \in C_c(\mathbb{R}^d; [0, \infty))$, we define

$$\int_{\mathbb{R}^d} f \, d\mu = \sup_{0 \leqslant s \leqslant f,\, s \text{ simple}} \int_{\mathbb{R}^d} s \, d\mu;$$

Step 3 For general real-valued function $f \in C_c(\mathbb{R}^d;\mathbb{R})$ we decompose it into its positive and negative parts:

$$f = f_+ - f_-, \text{ with } f_+ = f1_{\{f\geqslant 0\}},\ f_- = -f1_{\{f<0\}} \in C_c(\mathbb{R}^d;[0,\infty));$$

and for general complex-valued function $f \in C_c(\mathbb{R}^d;\mathbb{C})$ we separate its real and imaginary parts:

$$f = \mathrm{Re}\,[f] + i\mathrm{Im}\,[f].$$

In particular,

- $L^1_{\mathrm{loc}}(\mathbb{R}^d) \subset \mathcal{D}'(\mathbb{R}^d)$: Any locally Lebesgue integrable function f[51] defines a unique distribution $T_f \in \mathcal{D}'(\mathbb{R}^d)$ by

  $$\langle T_f, \varphi\rangle_{\mathcal{D}',\mathcal{D}} := T_f(\varphi) = \int_{\mathbb{R}^d} f\varphi dm^d,$$

  where m^d denotes the Lebesgue measure on $\mathbb{R}^d$.

  Thus all continuous functions and $L^p(\mathbb{R}^d)$-functions are distributions.

- The Dirac function $\delta_0 \in \mathcal{D}'(\mathbb{R}^d)$ defined by

  $$\langle \delta_0, \varphi\rangle_{\mathcal{D}',\mathcal{D}} = \delta_0(\varphi) = \varphi(0) \tag{2.43}$$

  is a distribution with compact support $\{0\}$.

  δ_0 defines a continuous linear nonnegative functional on $C_c(\mathbb{R}^d)$, and the definition coincides with the Lebesgue integral with respect to the Dirac measure: $\delta_0(\{0\}) = 1$ and $\delta_0(A) = 0$ if the Borel set $A \cap \{0\} = \varnothing$, as below

  $$\begin{aligned}\int_{\mathbb{R}^d} \varphi\, d\delta_0 &= \int_{\mathbb{R}^d} \varphi(0)1_{\{0\}}\, d\delta_0 + \int_{\mathbb{R}^d} \varphi 1_{\{0\}^C}\, d\delta_0 \\ &= \varphi(0)\delta_0(\{0\}) + 0 = \varphi(0).\end{aligned}$$

Products, derivatives and convolution Since

- The product between a smooth function and a test function is again a test function,
- Any derivative of a test function is again a test function,

[51] See Section 2.2.1 below for the definition.

- The convolution between two test functions is again a test function,

we can define by use of duality[52]

- The product between a smooth function $\phi \in C^\infty(\mathbb{R}^d)$ and a distribution $T \in \mathcal{D}'(\mathbb{R}^d)$ to be a distribution $\phi T \in \mathcal{D}'(\mathbb{R}^d)$, given by

$$\langle \phi T, \varphi \rangle_{\mathcal{D}',\mathcal{D}} = \langle T, \phi\varphi \rangle_{\mathcal{D}',\mathcal{D}}, \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^d);$$

- The α-derivative of $T \in \mathcal{D}'(\mathbb{R}^d)$ to be a distribution $\partial^\alpha T \in \mathcal{D}'(\mathbb{R}^d)$, given by

$$\langle \partial^\alpha T, \varphi \rangle_{\mathcal{D}',\mathcal{D}} = (-1)^{|\alpha|} \langle T, \partial^\alpha \varphi \rangle_{\mathcal{D}',\mathcal{D}}, \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^d);$$

- The convolution between a test function $\phi \in \mathcal{D}(\mathbb{R}^d)$ and a distribution $T \in \mathcal{D}'(\mathbb{R}^d)$ to be a distribution $\phi * T \in \mathcal{D}'(\mathbb{R}^d)$, given by

$$\langle \phi * T, \varphi \rangle_{\mathcal{D}',\mathcal{D}} = \langle T, \tilde{\phi} * \varphi \rangle_{\mathcal{D}',\mathcal{D}}, \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^d);$$

  with $\tilde{\phi}(x) = \phi(-x)$.

[Sep 29, 2026]

[Sep 30, 2026]

Note that the convolution $(\phi * T) \in \mathcal{D}'(\mathbb{R}^d)$ defined above is a smooth function:

$$(\phi * T)(x) = \langle T, \phi(x - \cdot) \rangle_{\mathcal{D}',\mathcal{D}}, \quad \forall x \in \mathbb{R}^d . \tag{2.44}$$

Indeed, if $x_n \to x$ in $\mathbb{R}^d$, then

$$\phi(x_n - \cdot) \to \phi(x - \cdot) \text{ in } \mathcal{D}(\mathbb{R}^d),$$

and thus by the continuity of T,

$$(\phi * T)(x_n) - (\phi * T)(x) = \langle T, \phi(x_n - \cdot) - \phi(x - \cdot) \rangle_{\mathcal{D}',\mathcal{D}} \to 0.$$

The continuity of the function $(\phi * T)(x)$ follows. Its smoothness follows from

$$\partial^\alpha (\phi * T)(x) = \langle T, (\partial^\alpha \phi)(x - \cdot) \rangle_{\mathcal{D}',\mathcal{D}}.$$

Therefore one can define the convolution between a distribution and a distribution with compact support[53]: For $T \in \mathcal{D}'(\mathbb{R}^d)$ and $S \in \mathcal{D}'(\mathbb{R}^d)$ with compact support,

$$\langle T * S, \varphi \rangle_{\mathcal{D}',\mathcal{D}} = \langle T, \tilde{S} * \varphi \rangle_{\mathcal{D}',\mathcal{D}}, \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^d), \tag{2.45}$$

where $\tilde{S}$ is defined as

$$\tilde{S}(\phi) = S(\tilde{\phi}), \quad \forall \phi \in \mathcal{D}(\mathbb{R}^d),$$

and $\tilde{S} * \varphi \in \mathcal{D}(\mathbb{R}^d)$.

[52] These definitions coincide with classical ones.

[53] In general, one can NOT define the product between two distributions.

Computations related to the Dirac function Recall the definition of the Dirac function in (2.43).

- Let $d = 1$, then the Dirac function δ_0 is the derivative of the Heaviside function: $H(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geqslant 0 \end{cases}$ which belongs to $L^\infty(\mathbb{R}) \subset L^1_{\text{loc}}(\mathbb{R}) \subset \mathcal{D}'(\mathbb{R})$. Indeed, for $\varphi \in \mathcal{D}(\mathbb{R})$,

$$\begin{aligned} (\partial_x H)(\varphi) &= (\partial_x T_H)(\varphi) = -T_H(\varphi') \\ &= -\int_0^\infty H(x)\varphi'(x)dx = \varphi(0) = \delta_0(\varphi). \end{aligned} \tag{2.46}$$

  One can further calculate $\partial_x \delta_0 \in \mathcal{D}'(\mathbb{R})$ as

$$(\partial_x \delta_0)(\varphi) = -\delta_0(\varphi') = -\varphi'(0), \quad \forall \varphi \in \mathcal{D}(\mathbb{R}),$$

  and iteratively $(\partial_x^n \delta_0)(\varphi) = (-1)^n \varphi^{(n)}(0)$.

- The Dirac function is the limit of the sequence (η_ε) in the sense of distribution:

$$\eta_\varepsilon \to \delta_0 \text{ in } \mathcal{D}'(\mathbb{R}^d),$$

  since for any $\varphi \in \mathcal{D}(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} \eta_\varepsilon(x)(\varphi(x) - \varphi(0))\,\mathrm{d}x = \int_{|x|<\varepsilon} \frac{1}{\varepsilon^d}\eta(\frac{x}{\varepsilon})\big(\varphi(x) - \varphi(0)\big)\,\mathrm{d}x \to 0;$$

  (η_ε) is called Dirac sequence.

- The convolution of some distribution $T \in \mathcal{D}'(\mathbb{R}^d)$ and the Dirac function is itself:

$$T * \delta_0 = T, \quad \forall T \in \mathcal{D}'(\mathbb{R}^d),$$

  since for any $\phi \in \mathcal{D}(\mathbb{R}^d)$,

$$(\phi * \delta_0)(x) = \langle \delta_0, \phi(x - \cdot)\rangle_{\mathcal{D}',\mathcal{D}} = \phi(x),$$

  and hence (2.45) implies

$$\langle \delta_0 * T, \varphi\rangle_{\mathcal{D}',\mathcal{D}} = \langle T, \widetilde{\delta}_0 * \varphi\rangle_{\mathcal{D}',\mathcal{D}} = \langle T, \varphi\rangle_{\mathcal{D}',\mathcal{D}}.$$

- Any distribution $T \in \mathcal{D}'(\mathbb{R}^d)$ can be approximated by a sequence of smooth functions:

$$\eta_\varepsilon * T \to \delta_0 * T = T \text{ in } \mathcal{D}'(\mathbb{R}^d),$$

since

$$\eta_\varepsilon * \varphi \to \varphi \text{ in } \mathcal{D}(\mathbb{R}^d).$$

Recall the approximation sequence (2.41), then any distribution $T \in \mathcal{D}'(\mathbb{R}^d)$ can be approximated by a sequence of test functions:

$$\eta_{2^{-n}} * (\phi_j T) \overset{j,n\to\infty}{\to} T \text{ in } \mathcal{D}'(\mathbb{R}^d). \tag{2.47}$$

This implies

$$\mathcal{D}(\mathbb{R}^d) \subset \mathcal{D}'(\mathbb{R}^d) \text{ is a dense subset.}$$

[Sep 30, 2026]
[Oct 13, 2026]

Smooth function space Note that a continuous function $f \in C(\mathbb{R}^d)$ is not necessarily bounded, and similarly for k-times continuously differentiable functions $f \in C^k(\mathbb{R}^d)$. Let $C^\infty(\mathbb{R}^d) = \cap_{k\in\mathbb{N}} C^k(\mathbb{R}^d)$, and by

$$\varphi_j \to \varphi \text{ in } C^\infty(\mathbb{R}^d)$$

we mean that for any compact subset $K \subset \mathbb{R}^d$ and any multiindex $\alpha = (\alpha_1, \cdots, \alpha_d)$, it holds

$$\sup{}_{x\in K} |\partial^\alpha(\varphi_j - \varphi)(x)| \to 0.$$

Let $\mathcal{E}'(\mathbb{R}^d)$ denote the space of all continuous linear map from $C^\infty(\mathbb{R}^d)$ to $\mathbb{K}$. As $\mathcal{D}(\mathbb{R}^d) \subset C^\infty(\mathbb{R}^d)$ has a stronger topology, $\mathcal{E}'(\mathbb{R}^d) \subset \mathcal{D}'(\mathbb{R}^d)$. Some distribution T belongs to $\mathcal{E}'(\mathbb{R}^d)$ if there exists some compact subset $K \subset \mathbb{R}^d$, some natural number $k \in \mathbb{N}$ and some positive constant C such that

$$|\langle T, \varphi\rangle| \leqslant C \sup{}_{x\in K, |\alpha|\leqslant k} |\partial^\alpha \varphi(x)|, \quad \forall \varphi \in C^\infty(\mathbb{R}^d).$$

Thus $\mathcal{E}'(\mathbb{R}^d)$ consists of distributions with compact support and as a typical example: $\delta_0 \in \mathcal{E}'(\mathbb{R}^d)$.

2.1.3 Schwartz functions and tempered distributions

Laurent Schwartz [54] "invented" Schwartz function space and its dual space – tempered distribution space, which turn out to be a perfect functional framework for the illustration of Fourier transform on $\mathbb{R}^d$.

Schwartz functions The Schwartz space $\mathcal{S}(\mathbb{R}^d)$ consists of smooth rapidly decaying functions. More precisely, $\mathcal{S}(\mathbb{R}^d)$ is the set of smooth functions $f \in C^\infty(\mathbb{R}^d)$ satisfying for any $k \in \mathbb{N}$

$$\|f\|_{k,\mathcal{S}} := \sup_{x\in\mathbb{R}^d,\,|\alpha|\leqslant k}(1+|x|)^k|\partial^\alpha f(x)| < \infty,$$

which is equivalent to say that

$$\mathcal{S}(\mathbb{R}^d) = \{f \in C^\infty(\mathbb{R}^d) \,|\, \sup_{x\in\mathbb{R}^d}|x^\alpha \partial^\beta f(x)| < \infty,\ \forall \text{ multiindices } \alpha,\beta\}.$$

Notice that for $f \in \mathcal{S}$, $\|f\|_{k,\mathcal{S}}$ may depend on $k \in \mathbb{N}$.
We say that

$$f_n \to f \text{ if } \mathcal{S} \text{ if } \|f_n - f\|_{k,\mathcal{S}} \to 0, \quad \forall k \in \mathbb{N}\,.$$

We can then introduce a metric $d_{\mathcal{S}}(\cdot,\cdot)$ on $\mathcal{S}$[55]:

$$d_{\mathcal{S}}(f,g) := \sum_{k\in\mathbb{N}} 2^{-k}\frac{\|f-g\|_{k,\mathcal{S}}}{1+\|f-g\|_{k,\mathcal{S}}}, \tag{2.48}$$

such that $d_{\mathcal{S}}(f_n, f) \to 0$ if and only if $f_n \to f$ in $\mathcal{S}$. The space $(\mathcal{S}(\mathbb{R}^d), d_{\mathcal{S}}(\cdot,\cdot))$ is a complete metric space and the space $\mathcal{D}(\mathbb{R}^d)$ is dense in it: The multiplication by the local approximation sequence of identity given in (2.41) approximates the function itself

$$d_{\mathcal{S}}(f, \phi_j f) \to 0 \text{ as } j \to \infty, \quad \forall f \in \mathcal{S}(\mathbb{R}^d).$$

It is also easy to see that

- $\mathcal{D}(\mathbb{R}^d) \subset \mathcal{S}(\mathbb{R}^d) \subset C_b^\infty(\mathbb{R}^d)$;
- If $f \in \mathcal{S}(\mathbb{R}^d)$ and $g \in C_b^\infty(\mathbb{R}^d)$, then $fg \in \mathcal{S}(\mathbb{R}^d)$;

[54] Laurent Schwartz, Généralisation de la notion de fonction, de dérivation, de transformation de Fourier et applications mathématiques et physiques, *Annales de l'Université de Grenoble* 21, 57–74, 1944/1945.

Laurent Schwartz, *Théorie des distributions*, Hermann, Paris, 1950–1951.

[55] The general idea of using a countable family of seminorms to define a metric was introduced by Maurice Fréchet in his 1906 doctoral thesis on metric spaces: *Sur quelques points du calcul fonctionnel*. Rendiconti del Circolo Matematico di Palermo, 22(1), 1–74.

- If $f \in \mathcal{S}(\mathbb{R}^d)$, then $x^\alpha f$, $\partial^\alpha f \in \mathcal{S}(\mathbb{R}^d)$ for any multiindex α;
- If $f, g \in \mathcal{S}(\mathbb{R}^d)$, then $f * g \in \mathcal{S}(\mathbb{R}^d)$;
- The Gaussian function $e^{-\frac{1}{2}|x|^2} \in \mathcal{S}(\mathbb{R}^d)$. The function $e^{-|x|} \notin \mathcal{S}(\mathbb{R}^d)$.

We can compute the integral of the Gaussian function (see here for a historical review of Gaussian function) as follows: For $d = 2$,

$$\int_{\mathbb{R}^2} e^{-\frac{1}{2}|x|^2} \,\mathrm{d}x = (2\pi) \int_0^\infty e^{-\frac{1}{2}r^2} r \, dr = (2\pi) \int_0^\infty e^{-\rho} \, d\rho = 2\pi.$$

Since $e^{-\frac{1}{2}|x|^2} = e^{-\frac{1}{2}x_1^2} \cdots e^{-\frac{1}{2}x_d^2}$, we deduce

$$\int_{\mathbb{R}^1} e^{-\frac{1}{2}x^2} \,\mathrm{d}x = \sqrt{2\pi}, \quad \int_{\mathbb{R}^d} e^{-\frac{1}{2}|x|^2} \,\mathrm{d}x = (2\pi)^{\frac{d}{2}}.$$

We rescale the Gaussian function $(2\pi)^{-\frac{d}{2}} e^{-\frac{1}{2}|x|^2}$ by ε to derive a Dirac sequence:

$$\left\{ p_\varepsilon(x) = \frac{1}{(2\pi)^{\frac{d}{2}}} \frac{1}{\varepsilon^d} e^{-\frac{1}{2}|\frac{x}{\varepsilon}|^2} \right\}_{\varepsilon > 0}, \quad x \in \mathbb{R}^d \,. \tag{2.49}$$

The probability density function (pdf) of a normally distributed random variable reads

$$\frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad x \in \mathbb{R},$$

with expected value μ and variance σ^2.

Tempered distribution A tempered distribution on $\mathbb{R}^d$ is a continuous linear map from $\mathcal{S}(\mathbb{R}^d)$ to $\mathbb{K}$. We denote the set of tempered distributions by $\mathcal{S}'(\mathbb{R}^d)$. We say that $T_j \to T$ in $\mathcal{S}'(\mathbb{R}^d)$ if

$$T_j(f) \to T(f) \quad \text{for any } f \in \mathcal{S}(\mathbb{R}^d),$$

that is, if there exists $k \in \mathbb{N}$ such that

$$\sup_{\|f\|_{k,\mathcal{S}} \leqslant 1} |T_j(f) - T(f)| \to 0.$$

Typical examples are

$$\delta_0, \quad x^\alpha, \text{ any polynomial on } \mathbb{R}^d \in \mathcal{S}'(\mathbb{R}^d),$$

but exponentially growing function $e^{|x|} \notin \mathcal{S}'(\mathbb{R}^d)$ (noticing $e^{|x|} \in L^1_{\mathrm{loc}}(\mathbb{R}^d) \subset \mathcal{D}'(\mathbb{R}^d)$).
Note that

- It holds the embedding

$$\mathcal{D}(\mathbb{R}^d) \subset \mathcal{S}(\mathbb{R}^d) \subset C_b^\infty(\mathbb{R}^d) \subset C_b(\mathbb{R}^d) \subset \mathcal{S}'(\mathbb{R}^d) \subset \mathcal{D}'(\mathbb{R}^d);$$

- We can define the product between $T \in \mathcal{S}'(\mathbb{R}^d)$ and $f \in C_b^\infty(\mathbb{R}^d) \cup \{\text{polynomials on } \mathbb{R}^d\}$ by

$$\langle fT, g\rangle_{\mathcal{S}',\mathcal{S}} = \langle T, fg\rangle_{\mathcal{S}',\mathcal{S}}, \quad \forall g \in \mathcal{S}(\mathbb{R}^d);$$

- We can define the derivative of $T \in \mathcal{S}'(\mathbb{R}^d)$ of any order by

$$\langle \partial^\alpha T, g\rangle_{\mathcal{S}',\mathcal{S}} = (-1)^{|\alpha|}\langle T, \partial^\alpha g\rangle_{\mathcal{S}',\mathcal{S}}, \quad \forall g \in \mathcal{S}(\mathbb{R}^d);$$

- If $f \in \mathcal{S}(\mathbb{R}^d)$ and $T \in \mathcal{S}'(\mathbb{R}^d)$, then $(f * T)(x) = \langle T, f(x - \cdot)\rangle_{\mathcal{S}',\mathcal{S}}$ is a smooth function and has tempered growth at infinity;

  Here, the smoothness follows as for (2.44), and tempered growth means polynomial growth at infinity, since there exists $k \in \mathbb{N}$ such that

$$\begin{aligned} &\big|\langle T, f(x-\cdot)\rangle_{\mathcal{S}',\mathcal{S}}\big| \leqslant C\big\|f(x-\cdot)\big\|_{k,\mathcal{S}} \\ &= C \sup_{y\in\mathbb{R}^d, |\alpha|\leqslant k} (1+|y|)^k |\partial^\alpha f(x-y)| \leqslant C(1+|x|)^k \|f\|_{k,\mathcal{S}}. \end{aligned}$$

[Oct 13, 2026]

[Oct 14, 2026]

- If $f \in \mathcal{S}(\mathbb{R}^d)$ and $T \in \mathcal{S}'(\mathbb{R}^d)$ with compact support (i.e. $T \in \mathcal{E}'(\mathbb{R}^d)$), then $f * T \in \mathcal{S}(\mathbb{R}^d)$;

- $\mathcal{S}(\mathbb{R}^d)$ is dense in $\mathcal{S}'(\mathbb{R}^d)$, since (recalling (2.49))

$$p_\varepsilon * (e^{-\frac{\delta}{2}|x|^2}\varphi) \to \varphi \text{ in } \mathcal{S}(\mathbb{R}^d), \text{ as } \varepsilon \to 0, \quad \delta \to 0, \tag{2.50}$$

  and correspondingly for $T \in \mathcal{S}'(\mathbb{R}^d)$,

$$\mathcal{S}(\mathbb{R}^d) \ni e^{-\frac{\delta}{2}|x|^2}\big(p_\varepsilon * T\big) \to T \text{ in } \mathcal{S}'(\mathbb{R}^d).$$

2.2 Lebesgue function framework

We endow the whole space $\mathbb{R}^d$, $d \geqslant 1$ with the Lebesgue measure m^d[56], which is the complete measure defined on the Lebesgue σ-algebra (as the completion of the Borel σ-algebra). In particular, the Lebesgue measure of a box

$$\Pi_{j=1}^d [a_j, b_j], \quad -\infty < a_j \leqslant b_j < \infty,$$

in $\mathbb{R}^d$ is $\Pi_{j=1}^d (b_j - a_j)$. It is a Radon measure, and defines a continuous linear positive functional on $C_c(\mathbb{R}^d)$:

$$C_c(\mathbb{R}^d) \ni f \mapsto \int_{\mathbb{R}^d} f \, dm^d =: \int_{\mathbb{R}^d} f \, \mathrm{d}x \in \mathbb{C}\,.$$

2.2.1 $L^1(\mathbb{R}^d)$-functions

A real-valued function $f : \mathbb{R}^d \mapsto \mathbb{R}$ is called Lebesgue measurable if for any $\alpha \in \mathbb{R}$ the set

$$\{x \in \mathbb{R}^d \mid f(x) > \alpha\}$$

is Lebesgue measurable. A complex-valued function $f : \mathbb{R}^d \mapsto \mathbb{C}$ is Lebesgue measurable if its real and imaginary parts are both Lebesgue measurable. We consider the Lebesgue integrable function space on $\mathbb{R}^d$:

$$L^1(\mathbb{R}^d; \mathbb{C}) = \Big\{ f : \mathbb{R}^d \to \mathbb{C} \text{ is Lebesgue measurable} \,\Big|\, \|f\|_{L^1(\mathbb{R}^d)} = \int_{\mathbb{R}^d} |f(x)| \, \mathrm{d}x < \infty \Big\} / \sim,$$

where the Lebesgue integral is defined as the Riemann integral:

$$\int_{\mathbb{R}^d} |f(x)| \, \mathrm{d}x = \int_0^\infty m^d(\{x \in \mathbb{R}^d \mid |f(x)| > \alpha\}) d\alpha,$$

and we call two measurable functions equivalent $f \sim g$ if $m^d(\{f \neq g\}) = 0$. Its local version is defined by

$$L^1_{\mathrm{loc}}(\mathbb{R}^d; \mathbb{C}) = \{ f : \mathbb{R}^d \to \mathbb{C} \text{ is Lebesgue measurable} \,|\, f \in L^1(K; \mathbb{C}) \text{ for any compact subset } K \subset \mathbb{R}^d \} / \sim,$$

where we endow K with the Lebesgue measure restricted to K, and

$$\|f\|_{L^1(K)} = \int_K |f(x)| \, \mathrm{d}x = \int_0^\infty m^d(\{x \in K \mid |f(x)| > \alpha\}) \, d\alpha.$$

[56] Henri Lebesgue, Leçons sur l'intégration et la recherche des fonctions primitives. Paris: Gauthier-Villars. 1902.

If $f, g \in L^1(\mathbb{R}^d;\mathbb{C})$, then the following functions are all in $L^1(\mathbb{R}^d;\mathbb{C})$:

- The translated function $\tau_{x_0}(f)$, defined by
$$\big(\tau_{x_0}(f)\big)(x) = f(x - x_0), \quad x_0 \in \mathbb{R}^d;$$
- The modulated function
$$e^{ix\cdot\xi_0} f(x), \quad \xi_0 \in \mathbb{R}^d;$$
- The rescaled function $f \circ A$, with $A \in \mathbb{R}^{d\times d}$ an invertible matrix, defined by
$$(f \circ A)(x) = f(Ax);$$
For example, if $A = \lambda \mathrm{Id}_{\mathbb{R}^{d\times d}}$ with $\lambda > 0$, then $(f \circ A)(x) = f(\lambda x)$.
- The convolution $f * g$
$$(f * g)(x) = \int_{\mathbb{R}^d} f(x-y)g(y)\,\mathrm{d}y = \int_{\mathbb{R}^d} f(y)g(x-y)\,\mathrm{d}y\,.$$

Density of $C_c(\mathbb{R}^d)$ in $L^1(\mathbb{R}^d)$ The Riemann integral coincides with the Lebesgue integral on $C_c(\mathbb{R}^d)$. We recall briefly the proof of the fact[57] that
$$C_c(\mathbb{R}^d) \subset L^1(\mathbb{R}^d), \text{ densely},$$
and thus $\overline{C_c(\mathbb{R}^d)}^{\|\|_{L^1}} = L^1(\mathbb{R}^d)$. The idea is as in the definition of the integral in (2.42). Firstly, since each complex-valued function f reads as $\mathrm{Re}\, f + i\mathrm{Im}\, f$, and each real-valued function f can be written as $f = f_+ - f_-$, where $f_+ = f1_{\{f\geqslant 0\}}$ and $f_- = -f1_{\{f<0\}}$, it suffices to consider nonnegative function f. Let $f \in L^1(\mathbb{R}^d)$ with $f \geqslant 0$. Since
$$\int_{\mathbb{R}^d} f dm^d = \int_0^\infty m^d(\{f > t\})dt,$$
given $\varepsilon > 0$ there exists $0 = t_0 < t_1 < \ldots t_j < t_{j+1} < t_N < \infty$ so that (with $t_0 = 0$)
$$0 < \int_0^\infty m^d(\{f > t\})dt - \sum_{j=1}^N (t_j - t_{j-1}) m^d(\{f > t_j\}) < \varepsilon.$$

[57]This density fact can also easily follow, provided that

- Any nonnegative measurable function can be approximated by a sequence of monotone nonnegative simple functions pointwise.
- Lebesgue dominate theorem.

Let

$$A_j = \{x : f(x) > t_j\}, \quad j = 1, \cdots, N,$$

such that

$$A_{j+1} \subset A_j \text{ and } m^d(A_1) < \infty.$$

Then

$$\Big\| f - \sum_{j=1}^{N} (t_j - t_{j-1}) 1_{A_j} \Big\|_{L^1} < \varepsilon.$$

Since there exists R such that $\int_{\{|x|>R\}} f dm^d < \varepsilon$, it suffices to approximate a characteristic function of a bounded measurable set A of finite measure by a continuous function.
Let $\varepsilon > 0$. By inner and outer regularity of the Lebesgue measure, there exists a compact set K and an open set U so that

$$K \subset A \subset U \qquad m^d(U) < m^d(K) + \varepsilon.$$

Let $d(K, \mathbb{R}^d \setminus U) =: d_0 > 0$ and we define

$$f_L(x) = \max\{1 - Ld(x, K), 0\} \in C(\mathbb{R}^d).$$

If $d_0 L \geqslant 1$, then

$$\|f_L - 1_A\|_{L^1} \leqslant \int_{U \setminus K} dm^d < \varepsilon.$$

If L is sufficiently large then Supp f_L is compact. Thus continuous functions with compact support are dense in $L^1(\mathbb{R}^d)$.

2.2.2 $L^p(\mathbb{R}^d)$-functions

A measurable function $f : \mathbb{R}^d \mapsto \mathbb{C}$ is called p-integrable, $p \in [1, \infty)$ if $|f|^p$ is integrable and we denote its $L^p(\mathbb{R}^d)$-norm as follows

$$\|f\|_{L^p(\mathbb{R}^d)} = \Big(\int_{\mathbb{R}^d} |f(x)|^p \, \mathrm{d}x \Big)^{1/p} = \Big(p \int_0^\infty \alpha^{p-1} m^d(\{x \in \mathbb{R}^d \mid |f(x)| > \alpha\}) d\alpha \Big)^{\frac{1}{p}}.$$

If $p = \infty$, then we say that the function f is essentially bounded, if it has finite L^∞-norm:

$$\|f\|_{L^\infty(\mathbb{R}^d)} = \inf_\alpha \big\{ \alpha \,|\, m^d(\{x \in \mathbb{R}^d \mid |f(x)| > \alpha\}) = 0 \big\}.$$

We define $L^p(\mathbb{R}^d)$ as the space of equivalence classes of p-integrable functions. The locally p-integral function space $L^p_{\text{loc}}(\mathbb{R}^d)$ consists of measurable functions which are p-integrable on any compact set in $\mathbb{R}^d$.
We recall here some basic facts concerning the Lebesgue spaces $L^p(\mathbb{R}^d)$, $p \in [1, \infty]$.

- The Lebesgue space $L^p(\mathbb{R}^d)$, $p \in [1,\infty]$ is Banach space. In particular, $L^2(\mathbb{R}^d)$ is Hilbert space with the inner product

$$\langle f, g\rangle_{L^2(\mathbb{R}^d)} = \int_{\mathbb{R}^d} f(x)\bar{g}(x)\,\mathrm{d}x\,.$$

- The pace $L^p(\mathbb{R}^d)$, $p \in [1,\infty)$ is separable, that is, there exists a countable dense subset in each $L^p(\mathbb{R}^d)$, $p \in [1,\infty)$. However, the Banach space $L^\infty(\mathbb{R}^d)$ is not separable.

- The test function set $\mathcal{D}(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$, $p \in [1,\infty)$. In particular, the convergence such as (2.47) holds in $L^p(\mathbb{R}^d)$. However, $\mathcal{D}(\mathbb{R}^d)$ is not dense in $L^\infty(\mathbb{R}^d)$.

  The inclusion $\mathcal{D}(\mathbb{R}^d) \subset \mathcal{S}(\mathbb{R}^d) \subset L^p(\mathbb{R}^d) \subset \mathcal{S}'(\mathbb{R}^d) \subset \mathcal{D}'(\mathbb{R}^d)$ holds.

- The Banach space $L^p(\mathbb{R}^d)$, $p \in (1,\infty)$ is reflexive, and its dual space is isomorphic to $L^{p'}(\mathbb{R}^d)$ with $\frac{1}{p} + \frac{1}{p'} = 1$. However, $L^1(\mathbb{R}^d)$ and $L^\infty(\mathbb{R}^d)$ are not reflexive: $(L^1)' = L^\infty$, $L^1 \subsetneq (L^\infty)'$[58].

- The Hölder's inequality holds for all $p \in [1,\infty]$: The product of two functions $f \in L^p$, $g \in L^{p'}$ lies in L^1, and furthermore[59]

$$\int_{\mathbb{R}^d} |fg|\,\mathrm{d}x \leqslant \|f\|_{L^p}\|g\|_{L^{p'}}. \tag{2.51}$$

  This implies immediately (noticing $|f|^q = (|f|^p)^{\frac{\theta q}{p}}(|f|^r)^{\frac{(1-\theta)q}{r}}$ and $1 = \frac{\theta q}{p} + \frac{(1-\theta)q}{r}$)

$$\|f\|_{L^q} \leqslant \|f\|_{L^p}^{\theta}\|f\|_{L^r}^{1-\theta} \quad \text{with } \frac{1}{q} = \frac{\theta}{p} + \frac{1-\theta}{r}, \quad p \leqslant q \leqslant r, \quad \theta \in [0,1],$$
$$\text{and hence } \|f\|_{L^q} \leqslant \|f\|_{L^p} + \|f\|_{L^r}. \tag{2.52}$$

- The Minkowski's inequality holds:

$$\|f+g\|_{L^p} \leqslant \|f\|_{L^p} + \|g\|_{L^p}, \quad p \in [1,\infty], \quad f, g \in L^p(\mathbb{R}^d).$$

- The Young's inequality holds:

$$\|f * g\|_{L^r(\mathbb{R}^d)} \leqslant \|f\|_{L^p(\mathbb{R}^d)}\|g\|_{L^q(\mathbb{R}^d)}, \quad 1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}, \quad p, q, r \in [1,\infty].$$

[58] δ_0, as a linear continuous functional defined on $C_c(\mathbb{R}^d)$, can be extended to an element in $(L^\infty)'$ by Hahn-Banach theorem. However, $\delta_0 \notin L^1(\mathbb{R}^d)$: if $\delta_0 \in L^1(\mathbb{R}^d)$, then for a sequence $\varphi_k \subset \mathcal{D}(\mathbb{R}^d)$ with $\varphi_k(0) = 1$ while $\varphi_k \xrightarrow{a.e.} 0$, Lebesgue dominated convergence theorem implies $\int_{\mathbb{R}^d} \delta_0\varphi_k\,\mathrm{d}x \to 0$ which contradicts the definition $\delta_0(\varphi_k) = 1$.

[59] The case $p = q = 2$ is referred to as Cauchy-Schwarz inequality.

2.2.3 Sobolev spaces

Recall the distribution theory introduced in Section 2.1.2. For $k \in \mathbb{N}$, $1 \leqslant p \leqslant \infty$, the Sobolev space[60] $W^{k,p}(\mathbb{R}^d) \subset L^p(\mathbb{R}^d)$ is the set of all $L^p(\mathbb{R}^d)$ functions, so that for all multiindices α of length at most k there exists $f_\alpha \in L^p(\mathbb{R}^d)$ so that

$$\partial^\alpha T_f = T_{f_\alpha}.$$

By an abuse of notation we identify $\partial^\alpha T_f$ with f_α, and hence $g = \partial_{x_j} f$ in $L^p(\mathbb{R}^d)$ for $f \in L^p(\mathbb{R}^d)$ means

$$\int_{\mathbb{R}^d} g\varphi dm^d = -\int_{\mathbb{R}^d} f\partial_{x_j}\varphi dm^d, \quad \forall \phi \in \mathcal{D}(\mathbb{R}^d). \tag{2.53}$$

We endow $W^{k,p}(\mathbb{R}^d)$ with the following norms:

$$\|f\|_{W^{k,p}(\mathbb{R}^d)} = \left(\sum_{|\alpha|\leqslant k} \|\partial^\alpha f\|^p_{L^p(\mathbb{R}^d)}\right)^{\frac{1}{p}}, \quad p \in [1,\infty),$$

and

$$\|f\|_{W^{k,\infty}(\mathbb{R}^d)} = \sup{}_{|\alpha|\leqslant k}\|\partial^\alpha f\|_{L^\infty(\mathbb{R}^d)}.$$

Then $(W^{k,p}(\mathbb{R}^d), \|\|_{W^{k,p}})$, $p \in [1,\infty]$ is a Banach space, and for $p \in [1,\infty)$,

$$\overline{\mathcal{D}(\mathbb{R}^d)}^{\|\|_{W^{k,p}}} = W^{k,p}(\mathbb{R}^d), \quad p \in [1,\infty).$$

One can check

- If $g \in C^k_b(\mathbb{R}^d)$ and $f \in W^{k,p}(\mathbb{R}^d)$, then $gf \in W^{k,p}(\mathbb{R}^d)$ and the Leibniz' rule holds;
- For $f \in W^{k,p}(\mathbb{R}^d)$ and $\phi : \mathbb{R}^d \to \mathbb{R}^d$ a C^k_b diffeomorphism, the chain rule holds and
$$\|f \circ \phi\|_{W^{k,p}(\mathbb{R}^d)} \leqslant C\|f\|_{W^{k,p}(\mathbb{R}^d)}. \tag{2.54}$$

For different values of p we have:

- For $p = 2$, we denote $H^k(\mathbb{R}^d) := W^{k,2}(\mathbb{R}^d)$, and we define
$$\langle f, g\rangle_{H^k(\mathbb{R}^d)} = \sum_{|\alpha|\leqslant k} \langle \partial^\alpha f, \partial^\alpha g\rangle_{L^2(\mathbb{R}^d)}.$$
The inner-product space $(H^k(\mathbb{R}^d), \langle,\rangle_{H^k(\mathbb{R}^d)})$ is Hilbert space.

[60] Sergei L'vovich Sobolev, Über die Randwertaufgabe einer Gleichung von elliptischem Typus. Izv. Akad. Nauk SSSR Ser. Mat., 1 (1936), 57–86. [Original Russian title: О краевой задаче для уравнения эллиптического типа. Известия Акад. Наук СССР, Сер. Математическая, т. 1, с. 57–86.]

- For $1 < p < \infty$,
  - For $k \in \mathbb{N}$, we define $W^{-k,p}(\mathbb{R}^d) = (W^{k,p'}(\mathbb{R}^d))'$, which is equipped with the norm defined by
    $$\|T\|_{W^{-k,p}(\mathbb{R}^d)} = \sup_{f \in W^{k,p'}(\mathbb{R}^d),\, \|f\|_{W^{k,p'}(\mathbb{R}^d)} \leqslant 1} |T(f)|.$$
    $W^{k,p}(\mathbb{R}^d)$ and $W^{-k,p}(\mathbb{R}^d)$ are both reflexive.
  - For $s \in (0,1)$, define Sobolev-Slobodeckij space[61] $W^{s,p}(\mathbb{R}^d)$ as the set of $L^p(\mathbb{R}^d)$ functions such that the following seminorm is finite
    $$[f]_{s,p} := \Big(\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|f(x) - f(y)|^p}{|x-y|^{sp+d}} \,\mathrm{d}x \,\mathrm{d}y \Big)^{\frac{1}{p}} < \infty.$$
    More generally, for $s > 0$ not an integer, let $k \in (s-1, s)$ be an integer, and
    $$W^{s,p}(\mathbb{R}^d) = \{ f \in W^{k,p}(\mathbb{R}^d) \mid \|f\|_{W^{s,p}} = \|f\|_{W^{k,p}} + \sup_{|\alpha|=k} [\partial^\alpha f]_{s-k,p} < \infty \}.$$
- For $p \in [1, d)$, there exists $C > 0$ depending on d, p such that
  $$\|f\|_{L^q(\mathbb{R}^d)} \leqslant C \|Df\|_{L^p(\mathbb{R}^d)}, \quad \forall f \in W^{1,p}(\mathbb{R}^d),$$
  with $\frac{1}{q} = \frac{1}{p} - \frac{1}{d}$.
- For $p > d$, we have Morrey's inequality for $\tilde{f} \in W^{1,p}(\mathbb{R}^d)$: Every point is a Lebesgue point and the canonical representative f is continuous, such that
  $$|f(x) - f(y)| \leqslant c|x-y|^{1-\frac{d}{p}} \|Df\|_{L^p(B_{|x-y|}(x))}. \quad \forall x, y \in \mathbb{R}^d.$$
  That is, f is Hölder continuous: $f \in C^{1-\frac{d}{p}}(\mathbb{R}^d)$.
- If $p = \infty$, then Lipschitz continuous functions $f \in W^{1,\infty}(\mathbb{R}^d)$ are almost everywhere differentiable.

[Oct 14, 2026]
[Oct 20, 2026]

[61] Nikolai Semenovich Slobodeckii, Estimates in the large of the solutions of certain boundary value problems. Dokl. Akad. Nauk SSSR, 118 (1958), 173 - 176. [Russian: Соболев, Н. С. Оценки решений краевых задач "в большом". ДАН СССР, т. 118, с. 173-176.]

It is also called Gagliardo-Slobodeckij space. Ennio Gagliardo, Estensioni di funzioni appartente a campi L_p con derivate in media quadratica. Rend. Sem. Mat. Univ. Padova, 25 (1956), 278-291.

3 Fourier transform

In this section we introduce the Fourier transform of functions defined in the whole space $\mathbb{R}^d$, $d \geqslant 1$.
Recall the Fourier series discussed in Section 1.2 for some function $f \in L^2([-\pi,\pi];\mathbb{C})$

$$f(x) = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{inx}, \text{ where } \hat{f}(n) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(x) e^{-inx} \, \mathrm{d}x \,. \quad (3.55)$$

Here the functions $\frac{1}{\sqrt{2\pi}} e^{inx}$, $n \in \mathbb{Z}$ are normalized eigenfunctions of the eigenvalue problem (1.12)

$$\mathcal{C}u = \lambda u, \quad \mathcal{C} : D(\mathcal{C}) \to Y, \quad \mathcal{C}(u) = -u'' \text{ for } u \in D(\mathcal{C})$$

associated to the eigenvalues n^2, where

- $Y = C([-\pi,\pi];\mathbb{C})$ is equipped with the inner product $\langle \cdot, \cdot \rangle_{L^2(-\pi,\pi)}$;
- $D(\mathcal{C}) = \{u \in C^2([-\pi,\pi]) \,|\, u(-\pi) = u(\pi), \quad u'(-\pi) = u'(\pi)\} \subset Y$.

Now, if we consider the free Schrödinger operator on the whole line

$$H_0 : H^2(\mathbb{R}) \to L^2(\mathbb{R}) \text{ via } H_0(f) = -f'',$$

then, unlike the operator $\mathcal{C}$ having discrete spectrum $\sigma(\mathcal{C}) = \mathbb{N} \cup \{0\}$, H_0 has continuous spectrum

$$\sigma(H_0) = [0, \infty).$$

Although the plane wave functions $\frac{1}{\sqrt{2\pi}} e^{ix\xi}$, $\xi \in \mathbb{R}$ satisfy

$$H_0(\frac{1}{\sqrt{2\pi}} e^{ix\xi}) = \xi^2 \cdot (\frac{1}{\sqrt{2\pi}} e^{ix\xi})$$

with $\xi^2 \in [0,\infty)$, they are not eigenfunctions of H_0 since they don't belong to $L^2(\mathbb{R})$: They are called generalized eigenfunctions. A similar representation as (3.55) in terms of the generalized eigenfunctions of the self-adjoint (unbounded) operator H_0 holds in the Hilbert space $L^2(\mathbb{R})$:

$$\begin{aligned} f(x) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f}(\xi) e^{ix\xi} \, \mathrm{d}\xi \,, \\ &\text{where } \hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-ix\xi} \, \mathrm{d}x \text{ in } L^2(\mathbb{R}). \end{aligned} \quad (3.56)$$

Its rigorous proof is found in Subsection 3.2.3 at the end of this chapter.

3.1 Fourier transform on Schwartz space $\mathcal{S}(\mathbb{R}^d)$

3.1.1 Fourier transform on $L^1(\mathbb{R}^d)$

Let $f = f(x) \in L^1(\mathbb{R}^d;\mathbb{C})$, $d \geqslant 1$. We define its Fourier transform as a function $\hat{f} = \hat{f}(\xi) \in L^\infty(\mathbb{R}^d;\mathbb{C})$ below

$$\hat{f}(\xi) := \mathcal{F}(f)(\xi) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x)\,\mathrm{d}x\,, \quad \forall \xi \in \mathbb{R}^d\,.$$

Observing the continuity of the function in ξ: $e^{-ix\cdot\xi}$ as well as the fact

$$-\xi_j e^{-ix\cdot\xi} = \frac{1}{i}\partial_{x_j}(e^{-ix\cdot\xi}), \quad j = 1, \cdots, d, \tag{3.57}$$

we derive the Riemann-Lebesgue lemma which states that $\hat{f}$ is continuous and satisfies[62]

$$\lim_{|\xi|\to\infty} |\hat{f}(\xi)| = 0.$$

Indeed, the continuity of $\hat{f}(\xi)$ follows from the continuity of the function in ξ: $e^{-ix\cdot\xi}$ and the Lebesgue convergence theorem, while the decay at infinity of $\hat{f}(\xi)$ follows from the density argument and integration by parts: If $f \in \mathcal{D}(\mathbb{R}^d)$, then $|\xi|\hat{f}(\xi)$ is bounded on $\mathbb{R}^d$ since

$$\xi_j \hat{f}(\xi) = \int_{\mathbb{R}^d} (-\frac{1}{i}\partial_{x_j})(e^{-ix\cdot\xi}) f(x)\,\mathrm{d}x \,= \int_{\mathbb{R}^d} e^{-ix\cdot\xi}(\frac{1}{i}\partial_{x_j}) f(x)\,\mathrm{d}x\,, \quad j = 1, \cdots, d.$$

Therefore the Fourier transform

$$\mathcal{F} : L^1(\mathbb{R}^d) \mapsto C_0(\mathbb{R}^d) \subset C_b(\mathbb{R}^d)$$

is a linear continuous map and

$$\|\mathcal{F}(f)\|_{L^\infty(\mathbb{R}^d)} \leqslant \frac{1}{(2\pi)^{\frac{d}{2}}} \|f\|_{L^1(\mathbb{R}^d)}, \text{ i.e. } \|\mathcal{F}\|_{L^1(\mathbb{R}^d)\mapsto L^\infty(\mathbb{R}^d)} \leqslant \frac{1}{(2\pi)^{\frac{d}{2}}}.$$

The action introduced in Subsection 2.2.1 behaves as follows after Fourier transform:

- $\mathcal{F}(\tau_{x_0}(f)) = e^{-ix_0\cdot\xi}\mathcal{F}(f)$, $x_0 \in \mathbb{R}^d$;
- $\mathcal{F}(e^{ix\cdot\xi_0} f) = \tau_{\xi_0}\mathcal{F}(f)$, $\xi_0 \in \mathbb{R}^d$;
- $\mathcal{F}(f \circ A) = |\det A|^{-1}\mathcal{F}(f) \circ A^{-T}$, $A \in \mathbb{R}^{d\times d}$ invertible, and in particular
$$\mathcal{F}(f(\lambda\cdot)) = \lambda^{-d}(\mathcal{F}(f))(\lambda^{-1}\cdot),\ \forall \lambda > 0;$$
- $\mathcal{F}(f * g) = (2\pi)^{\frac{d}{2}}\mathcal{F}(f)\mathcal{F}(g)$, $f, g \in L^1(\mathbb{R}^d;\mathbb{C})$;
- $\int_{\mathbb{R}^d} f\mathcal{F}(g)\,\mathrm{d}x \,= \int_{\mathbb{R}^d} \mathcal{F}(f) g\,\mathrm{d}x\,$, $f, g \in L^1(\mathbb{R}^d;\mathbb{C})$.

[62] Recall a similar vanishing property (1.18) in the case of interval $[-\pi, \pi]$.

3.1.2 Fourier transform on $\mathcal{S}(\mathbb{R}^d)$

Since $\mathcal{S}(\mathbb{R}^d) \subset L^1(\mathbb{R}^d)$, we can define the Fourier transform on $\mathcal{S}(\mathbb{R}^d)$. We claim that the Fourier transform

$$\mathcal{F} : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$$

is continuous:

- For any integer $k \in \mathbb{N}$, there exist a constant C and an integer $N \in \mathbb{N}$ such that

$$\|\mathcal{F}(f)\|_{k,\mathcal{S}} \leqslant C\|f\|_{N,\mathcal{S}}, \quad \forall f \in \mathcal{S}(\mathbb{R}^d).$$

- $\mathcal{F}$ maps the differentiation operator $(\frac{1}{i}\partial_x)^\alpha$ to the multiplication operator ξ^α, and vice versa (up to $(-1)^{|\alpha|}$):

$$\mathcal{F}((\frac{1}{i}\partial_x)^\alpha f) = \xi^\alpha \mathcal{F}(f), \quad \mathcal{F}(x^\alpha f) = (i\partial_\xi)^\alpha \mathcal{F}(f), \quad \forall \text{multiindex } \alpha. \tag{3.58}$$

Indeed, noticing the general version of (3.57)

$$(\frac{1}{i}\partial_x)^\alpha (e^{-ix\cdot\xi}) = (-1)^{|\alpha|}\xi^\alpha e^{-ix\cdot\xi}, \quad (\frac{1}{i}x)^\alpha e^{-ix\cdot\xi} = \partial_\xi^\alpha (e^{-ix\cdot\xi}),$$

it is straightforward to use Lebesgue's theorem and integration by parts to show (3.58). Then for any $|\alpha| + |\beta| \leqslant k$, there exist C and N (we can simply take $N = k + d + 1$) such that

$$\begin{aligned}\left|\xi^\alpha \partial_\xi^\beta \mathcal{F}(f)\right| &= \left|\mathcal{F}\big((\frac{1}{i}\partial_x)^\alpha (\frac{1}{i}x)^\beta f\big)\right| \leqslant (2\pi)^{-\frac{d}{2}} \|\partial_x^\alpha (x^\beta f)\|_{L^1} \\ &\leqslant \|(1+|x|)^{-d-1}\|_{L^1} \|(1+|x|)^{d+1}\partial_x^\alpha (x^\beta f)\|_{L^\infty} \leqslant C\|f\|_{N,\mathcal{S}}.\end{aligned}$$

Fourier transform of Gaussian function As an example, we compute the Fourier transform of the Gaussian function, which turns out to be itself:

$$\mathcal{F}(e^{-\frac{1}{2}|x|^2}) = e^{-\frac{1}{2}|\xi|^2}. \tag{3.59}$$

Indeed, we first notice that the Gaussian function satisfies the following equation

$$(\partial_{x_j} + x_j) e^{-\frac{1}{2}|x|^2} = 0, \quad j = 1, \cdots, d,$$

and hence by (3.58),

$$i(\xi_j + \partial_{\xi_j})\mathcal{F}(e^{-\frac{1}{2}|x|^2}) = 0, \quad j = 1, \cdots, d.$$

Let $d = 1$, then the function $\phi(\xi) := \mathcal{F}(e^{-\frac{1}{2}|x|^2})(\xi) \in \mathcal{S}(\mathbb{R})$ satisfies the following first-order differential equation

$$\phi' + \xi\phi = 0, \text{ i.e. } (e^{\frac{1}{2}\xi^2}\phi)' = 0,$$

and hence there exists a constant $C \in \mathbb{R}$ such that

$$\phi(\xi) = Ce^{-\frac{1}{2}\xi^2}.$$

In particular,

$$C = \phi(0) = \mathcal{F}(e^{-\frac{1}{2}x^2})(0) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{1}{2}x^2} \, \mathrm{d}x \, = 1,$$

and thus

$$\mathcal{F}(e^{-\frac{1}{2}x^2}) = \phi(\xi) = e^{-\frac{1}{2}\xi^2}.$$

For $d \geqslant 2$, we simply notice that

$$\begin{aligned}
\mathcal{F}(e^{-\frac{1}{2}|x|^2}) &= \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} e^{-\frac{1}{2}|x|^2} \, \mathrm{d}x \\
&= \frac{1}{(2\pi)^{\frac{d-1}{2}}} \int_{\mathbb{R}^{d-1}} e^{-ix'\cdot\xi'} e^{-\frac{1}{2}|x'|^2} \, \mathrm{d}x' \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{\mathbb{R}} e^{-ix_d\cdot\xi_d} e^{-\frac{1}{2}(x_d)^2} \, \mathrm{d}x_d \\
&= \frac{1}{(2\pi)^{\frac{d-1}{2}}} \int_{\mathbb{R}^{d-1}} e^{-ix'\cdot\xi'} e^{-\frac{1}{2}|x'|^2} \, \mathrm{d}x' e^{-\frac{1}{2}(\xi_d)^2} = \mathcal{F}(e^{-\frac{1}{2}|x'|^2}) e^{-\frac{1}{2}(\xi_d)^2},
\end{aligned}$$

with $x' = (x_1, \cdots, x_{d-1})$, $\xi' = (\xi_1, \cdots, \xi_{d-1})$. An easy induction argument implies the result for $d \geqslant 2$.

[Oct 20, 2026]
[Oct 21, 2026]

By the scaling property, we have

$$\mathcal{F}(e^{-\frac{\varepsilon^2}{2}|x|^2}) = \varepsilon^{-d} e^{-\frac{1}{2\varepsilon^2}|\xi|^2}. \tag{3.60}$$

The rescaled Gaussian function sequence $\{p_\varepsilon = (2\pi)^{-\frac{d}{2}} \varepsilon^{-d} e^{-\frac{1}{2\varepsilon^2}|\xi|^2}\}_\varepsilon$ is a Dirac sequence given in (2.49).

3.1.3 Inverse Fourier transform on $\mathcal{S}(\mathbb{R}^d)$

Let $f \in \mathcal{S}(\mathbb{R}^d)$. We define its inverse Fourier transform by

$$\check{f}(x) := \mathcal{F}^{-1}(f)(x) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{ix\cdot\xi} f(\xi)\,\mathrm{d}\xi\,.$$

Then the relation between Fourier transform and inverse Fourier transform follows immediately:

$$\begin{aligned}&\mathcal{F}^{-1}(f)(x) = \mathcal{F}(f)(-x) = \mathcal{F}(f(-\cdot))(x),\\ &\text{that is, } \mathcal{F}^{-1} = R\circ\mathcal{F} = \mathcal{F}\circ R,\end{aligned}$$

with R denoting the flip operator $(Rf)(x) = f(-x)$ on $\mathcal{S}(\mathbb{R}^d)$.
We claim that

$$\begin{aligned}&\mathcal{F}^{-1}\mathcal{F}(f)(x) = f(x) = \mathcal{F}\mathcal{F}^{-1}(f)(x),\\ &\text{that is, } \mathcal{F}^{-1}\mathcal{F} = \mathcal{F}\mathcal{F}^{-1} = \mathrm{Id} \text{ and } \mathcal{F}^2 = R \text{ on } \mathcal{S}(\mathbb{R}^d),\end{aligned} \tag{3.61}$$

and hence Fourier transform is an automorphism on $\mathcal{S}(\mathbb{R}^d)$. Indeed, for $f \in \mathcal{S}(\mathbb{R}^d)$ such that $\hat{f} \in \mathcal{S}(\mathbb{R}^d)$, by use of the pointwise convergence $\lim_{\varepsilon\to 0} e^{-\frac{\varepsilon^2}{2}|x|^2} = 1$, we calculate straightforwardly by definition that

$$\begin{aligned}\mathcal{F}^{-1}\mathcal{F}(f)(x) &= (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{ix\cdot\xi}\hat{f}(\xi)\,\mathrm{d}\xi \;= \lim_{\varepsilon\to 0}(2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-\frac{\varepsilon^2}{2}|\xi|^2} e^{ix\cdot\xi}\hat{f}(\xi)\,\mathrm{d}\xi\\ &= \lim_{\varepsilon\to 0}(2\pi)^{-d} \int_{\mathbb{R}^d} e^{-\frac{\varepsilon^2}{2}|\xi|^2} e^{ix\cdot\xi} \int_{\mathbb{R}^d} e^{-iy\cdot\xi} f(y)\,\mathrm{d}y\;\mathrm{d}\xi\\ &= \lim_{\varepsilon\to 0}(2\pi)^{-d} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{-\frac{\varepsilon^2}{2}|\xi|^2} e^{-i(y-x)\cdot\xi}\,\mathrm{d}\xi\, f(y)\,\mathrm{d}y\\ &= \lim_{\varepsilon\to 0}(2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} \mathcal{F}(e^{-\frac{\varepsilon^2}{2}|\xi|^2})(y-x) f(y)\,\mathrm{d}y\\ &= \lim_{\varepsilon\to 0}(2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} \varepsilon^{-d} e^{-\frac{1}{2\varepsilon^2}|y-x|^2} f(y)\,\mathrm{d}y \;= \lim_{\varepsilon\to 0} p_\varepsilon * f = f,\end{aligned} \tag{3.62}$$

where p_ε is the Dirac sequence given by (2.49), and we used (3.60). Hence $\mathcal{F}^{-1}\mathcal{F}(f)(x) = f(x)$ and $\mathcal{F}\mathcal{F}^{-1} = \mathrm{Id}$ follows simply from $\mathcal{F}^{-1}(f)(x) = \mathcal{F}(f)(-x)$.
From this we note

- It follows from

$$\mathcal{F}(f * g) = (2\pi)^{\frac{d}{2}}\mathcal{F}(f)\mathcal{F}(g), \quad f, g \in \mathcal{S}(\mathbb{R}^d)$$

that

$$\mathcal{F}(fg) = (2\pi)^{-\frac{d}{2}}\mathcal{F}(f) * \mathcal{F}(g), \quad f, g \in \mathcal{S}(\mathbb{R}^d).$$

In particular, it follows from

$$e^{-\frac{\varepsilon^2}{2}|x|^2} f \to f \text{ in } \mathcal{S}(\mathbb{R}^d) \text{ as } \varepsilon \to 0,$$

that

$$(2\pi)^{-\frac{d}{2}}\varepsilon^{-d}e^{-\frac{1}{2\varepsilon^2}|x|^2} * f \to f \text{ as } \mathcal{S}(\mathbb{R}^d) \text{ as } \varepsilon \to 0.$$

We revisit the density result (2.50).

- It follows from $\int_{\mathbb{R}^d} f\mathcal{F}(g)\,\mathrm{d}x = \int_{\mathbb{R}^d} \mathcal{F}(f)g\,\mathrm{d}x$ and $\overline{\mathcal{F}(g)} = \mathcal{F}^{-1}(\bar{g})$ the Parseval's identity[63]:

$$\int_{\mathbb{R}^d} f\bar{g}\,\mathrm{d}x = \int_{\mathbb{R}^d} f\mathcal{F}(\mathcal{F}^{-1}(\bar{g}))\,\mathrm{d}x = \int_{\mathbb{R}^d} \mathcal{F}(f)\,\mathcal{F}^{-1}(\bar{g}) = \int_{\mathbb{R}^d} \hat{f}\bar{\hat{g}}\,\mathrm{d}\xi\,. \tag{3.63}$$

In particular, the Plancherel's identity[64] holds:

$$\|f\|_{L^2} = \|\hat{f}\|_{L^2}. \tag{3.64}$$

3.2 Fourier transform on tempered distribution space $\mathcal{S}'(\mathbb{R}^d)$

3.2.1 Fourier transform on $\mathcal{S}'(\mathbb{R}^d)$

If $A : \mathcal{S}(\mathbb{R}^d) \mapsto \mathcal{S}(\mathbb{R}^d)$ is a linear operator, then by duality we can define the operator $A^t : \mathcal{S}'(\mathbb{R}^d) \mapsto \mathcal{S}'(\mathbb{R}^d)$ as follows:

$$(A^tT)(f) = T(Af), \quad \forall T \in \mathcal{S}'(\mathbb{R}^d), \quad f \in \mathcal{S}(\mathbb{R}^d).$$

We have seen such definitions in Subsection 2.1.3:

- The multiplication by $x_j : \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$, defined by

$$(x_jT)(f) = T(x_jf);$$

[63]Parseval, Marc-Antoine. "Mémoire sur les séries et sur l'intégration complète d'une équation aux différences partielles linéaires du second ordre, à coefficients constants." Mémoires présentés à l'Institut des Sciences, Lettres et Arts, par divers savans, et lus dans ses assemblées. Sciences, mathématiques et physiques. (Savans étrangers.), vol. 1, pp. 638–648. Presented 5 April 1799; published 1806.

[64]Michel Plancherel, Contribution à l'étude de la représentation d'une fonction arbitraire par des intégrales définies, *Rendiconti del Circolo Matematico di Palermo* 30, 289–335, 1910.

- The derivative $\partial_{x_j} : \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$, defined by

$$(\partial_{x_j} T)(f) = -T(\partial_{x_j} f);$$

- The convolution with $\mathcal{S}(\mathbb{R}^d) \ni g : \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$, defined by

$$(T * g)(x) = T(g(x - \cdot)) \in C^\infty \text{ has tempered growth.}$$

We further define the following operators on $\mathcal{S}'(\mathbb{R}^d)$:

- The translation $\tau_{x_0} : \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$, $x_0 \in \mathbb{R}^d$, defined by

$$(\tau_{x_0} T)(f) = T(\tau_{-x_0} f);$$

- The modulation $e^{ix\cdot\xi_0} : \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$, $\xi_0 \in \mathbb{R}^d$, defined by

$$(e^{ix\cdot\xi_0} T)(f) = T(e^{ix\cdot\xi_0} f);$$

- The rescaling $\circ A : \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$, $A \in \mathbb{R}^{d\times d}$ invertible matrix, defined by

$$(T \circ A)(f) = |\det(A)|^{-1} T(f \circ A^{-1});$$

- The Fourier transform $\mathcal{F} : \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$, defined by

$$\mathcal{F}(T)(f) = T(\mathcal{F}(f));$$

- The inverse Fourier transform $\mathcal{F}^{-1} : \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$, defined by

$$\mathcal{F}^{-1}(T)(f) = T(\mathcal{F}^{-1}(f)) = T(\mathcal{F}R(f)).$$

By virtue of (3.61), the Fourier transform is also an automorphism on $\mathcal{S}'(\mathbb{R}^d)$ with

$$\mathcal{F}^{-1}\mathcal{F} = \mathrm{Id} = \mathcal{F}\mathcal{F}^{-1}, \quad \mathcal{F}^{-1} = \mathcal{F} \circ R = R \circ \mathcal{F}, \quad \mathcal{F}^2 = R, \quad \text{on } \mathcal{S}'(\mathbb{R}^d),$$

and the following rules are valid on $\mathcal{S}'(\mathbb{R}^d)$:

$$\mathcal{F}(\tau_{x_0}(T)) = e^{-ix_0\cdot\xi}\mathcal{F}(T),\ \mathcal{F}(e^{ix\cdot\xi_0}T) = \tau_{\xi_0}\mathcal{F}(T), \quad \forall x_0, \xi_0 \in \mathbb{R}^d,$$
$$\mathcal{F}(T \circ A) = |\det A|^{-1}\mathcal{F}(T) \circ A^{-T},\ \text{e.g. } \mathcal{F}(T(\lambda\cdot)) = \lambda^{-d}(\mathcal{F}(T))(\lambda^{-1}\cdot),\ \forall \lambda > 0,$$
$$\mathcal{F}(T * f) = (2\pi)^{\frac{d}{2}}\mathcal{F}(T)\mathcal{F}(f),\ \mathcal{F}(Tf) = (2\pi)^{-\frac{d}{2}}\mathcal{F}(T) * \mathcal{F}(f), \quad \forall f \in \mathcal{S}(\mathbb{R}^d),$$
$$\mathcal{F}((\frac{1}{i}\partial_x)^\alpha T) = \xi^\alpha \mathcal{F}(T),\ \mathcal{F}(x^\alpha T) = (i\partial_\xi)^\alpha \mathcal{F}(T), \quad \forall \text{multiindex } \alpha.$$

3.2.2 Some typical examples

Example 3.1. *When $d = 1$, then we compute straightforwardly*

$$\begin{aligned}\mathcal{F}(e^{-|x|}) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\xi} e^{-|x|} \,\mathrm{d}x \\ &= \frac{1}{\sqrt{2\pi}} \Big(\int_0^\infty e^{-(1+i\xi)x} \,\mathrm{d}x \, + \int_{-\infty}^0 e^{(1-i\xi)x} \,\mathrm{d}x \Big) \\ &= \frac{1}{\sqrt{2\pi}} \Big(\frac{1}{1+i\xi} + \frac{1}{1-i\xi} \Big) = \sqrt{\frac{2}{\pi}} \frac{1}{1+\xi^2}. \end{aligned} \tag{3.65}$$

Example 3.2. *Let $d = 1$, and we compute the Fourier transform of the rectangular pulse/box function*

$$rect(x) = \begin{cases} 1, & |x| \leqslant T/2, \\ 0, & |x| > T/2. \end{cases}$$

as

$$\mathcal{F}(rect)(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-T/2}^{T/2} e^{-ix\xi} \,\mathrm{d}x \, = \frac{1}{\sqrt{2\pi}} \frac{1}{-i\xi} e^{-ix\xi} \Big|_{-\frac{T}{2}}^{\frac{T}{2}} = \sqrt{\frac{2}{\pi}} \frac{\sin(\frac{T}{2}\xi)}{\xi}, \quad \xi \in \mathbb{R}\,. \tag{3.66}$$

The sinc function $sinc(t) = \frac{\sin(t)}{t}$ has countably many zeros $\pm\pi, \pm 2\pi, \cdots$, and decays at infinity: Its oscillation cancels out when integrating

$$\int_{-\infty}^{\infty} \frac{\sin(\omega t)}{t} \,\mathrm{d}t \, = \int_{-\infty}^{\infty} \frac{\sin t}{t} \,\mathrm{d}t \, = 2 \int_0^\infty \frac{\sin t}{t} \,\mathrm{d}t \, = \pi, \quad \forall \omega > 0.$$

We prove this fact by use of Laplace transform[65] *of the sine function $f(t) = \sin t$ restricted on the positive half-line: For $s > 0$, define*

$$F(s) := \mathcal{L}(f)(s) = \int_0^\infty f(t) e^{-st} \,\mathrm{d}t \, = \int_0^\infty \frac{1}{2i} (e^{it} - e^{-it}) e^{-st} \,\mathrm{d}t$$

[65]Let $d = 1$. Laplace transform generalizes Fourier transform by replacing $i\xi \in i\,\mathbb{R}$ (which causes the oscillation in the Fourier kernel $e^{ix\xi}$) by some complex variable $s \in \mathbb{C}$. It transforms a function defined on $[0, \infty)$ to some function defined on some domain of $\mathbb{C}$ (usually on some right half plane with $\mathrm{Re}\, s > s_0$):

$$\mathcal{L}(f)(s) = \int_0^\infty f(t) e^{-st} \,\mathrm{d}t\,, \quad \mathrm{Re}\, s > s_0.$$

Even if $f(t)$ has exponential growth, say $f(t) = e^{at}$ with some $a > 0$, $\mathcal{L}(f)(s)$ is well-defined for $s \in \mathbb{C}$ with $\mathrm{Re}\, s > a$.

$$= \frac{1}{2i}\Big(\frac{1}{s-i} - \frac{1}{s+i}\Big) = \frac{1}{1+s^2}.$$

Thus

$$\begin{aligned}\int_0^\infty \frac{f(t)}{t}\,\mathrm{d}t &= \int_0^\infty f(t)\Big(\int_0^\infty e^{-st}ds\Big)\,\mathrm{d}t = \int_0^\infty\int_0^\infty f(t)e^{-st}\,\mathrm{d}t\,ds \\ &= \int_0^\infty F(s)ds = \int_0^\infty \frac{1}{1+s^2}ds = \frac{\pi}{2}.\end{aligned}$$

Example 3.3. *Let $d = 1$. Let $T = \text{p.v.}\,(\frac{1}{\pi}\frac{1}{x})$, with* p.v. *denoting the Cauchy principal value, be the tempered distribution defined by*

$$T(\varphi) = \text{p.v.}\,\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1}{x}\varphi(x)\,\mathrm{d}x := \frac{1}{\pi}\lim_{\varepsilon\to 0}\int_{\varepsilon<|x|<\frac{1}{\varepsilon}}\frac{1}{x}\varphi(x)\,\mathrm{d}x\,.$$

Then for any Schwartz function $\varphi \in \mathcal{S}(\mathbb{R})$,

$$\begin{aligned}\langle\hat{T},\varphi\rangle_{\mathcal{S}',\mathcal{S}} = \langle T,\hat{\varphi}\rangle_{\mathcal{S}',\mathcal{S}} &= \frac{1}{\pi}\lim_{\varepsilon\to 0}\int_{\varepsilon<|\xi|<\frac{1}{\varepsilon}}\frac{1}{\xi}\hat{\varphi}(\xi)\,\mathrm{d}\xi \\ &= \frac{1}{\pi}\lim_{\varepsilon\to 0}\int_{\frac{1}{\varepsilon}>|\xi|>\varepsilon}\frac{1}{\xi}\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\varphi(x)e^{-ix\xi}\,\mathrm{d}x\;\mathrm{d}\xi \\ &= \frac{1}{\pi}\frac{1}{\sqrt{2\pi}}\lim_{\varepsilon\to 0}\int_{\mathbb{R}}\varphi(x)\underbrace{\int_{\frac{1}{\varepsilon}>|\xi|>\varepsilon}\frac{1}{\xi}e^{-ix\xi}\,\mathrm{d}\xi}_{=\int_{\frac{1}{\varepsilon}>|\xi|>\varepsilon}\frac{-i\sin(x\xi)}{\xi}\,\mathrm{d}\xi}\;\mathrm{d}x \\ &= \frac{1}{\pi}\frac{1}{\sqrt{2\pi}}\lim_{\varepsilon\to 0}\int_{\mathbb{R}}\varphi(x)(-i\mathrm{sgn}\,(x))\underbrace{\int_{\frac{1}{\varepsilon}>|\xi|>\varepsilon}\frac{\sin(|x|\xi)}{\xi}\,\mathrm{d}\xi}_{\to\pi\ \text{as}\ \varepsilon\to 0}\;\mathrm{d}x \\ &= \Big\langle\frac{-i\mathrm{sgn}\,(x)}{\sqrt{2\pi}},\varphi\Big\rangle_{\mathcal{S}',\mathcal{S}}.\end{aligned}$$

Hence

$$\mathcal{F}(\text{p.v.}\,(\frac{1}{\pi}\frac{1}{x})) = \frac{1}{\sqrt{2\pi}}(\frac{1}{i}\mathrm{sgn}\,(\xi)). \tag{3.67}$$

[Oct 21, 2026]
[Oct 27, 2026]

Let us explain the well-definedness of $T = \text{p.v.}\,(\frac{1}{\pi}\frac{1}{x}) \in \mathcal{S}'(\mathbb{R})$: For $\varphi \in \mathcal{S}(\mathbb{R})$,

$$\Big\langle\text{p.v.}\,(\frac{1}{\pi}\frac{1}{x}),\varphi\Big\rangle_{\mathcal{S}',\mathcal{S}} = \lim_{\varepsilon\to 0}\frac{1}{\pi}\Big(\int_{\varepsilon}^{\infty} + \int_{-\infty}^{-\varepsilon}\Big)\frac{1}{x}\varphi(x)\,\mathrm{d}x\,,\ \text{since}\ \varphi\in\mathcal{S}(\mathbb{R}),$$

$$= \lim_{\varepsilon\to 0}\frac{1}{\pi}\int_{\varepsilon}^{\infty}\frac{1}{x}\big(\varphi(x)-\varphi(-x)\big)\,\mathrm{d}x\,,\ \text{by change of variables,}$$
$$= \frac{1}{\pi}\int_{0}^{\infty}\frac{1}{x}\big(\varphi(x)-\varphi(-x)\big)\,\mathrm{d}x\,,\ \text{by the smoothness of } \varphi. \tag{3.68}$$

Example 3.4 (Fourier transform of Dirac function)**.** *The (inverse) Fourier transform of the Dirac function is a constant function:*

$$\mathcal{F}^{-1}(\delta) = \mathcal{F}(\delta) = (2\pi)^{-\frac{d}{2}},$$

where $\delta := \delta_0 \in \mathcal{S}'(\mathbb{R}^d)$*, and* $(2\pi)^{-\frac{d}{2}} \in \mathcal{S}'(\mathbb{R}^d)$ *is a constant function defined as tempered distribution by*

$$T_{(2\pi)^{-\frac{d}{2}}}(f) = \int_{\mathbb{R}^d}(2\pi)^{-\frac{d}{2}} f(x)\,\mathrm{d}x\,, \quad \forall f \in \mathcal{S}(\mathbb{R}^d).$$

Indeed, it is easy to calculate

$$\hat{\delta}(f) = \delta(\hat{f}) = \hat{f}(0) = (2\pi)^{-\frac{d}{2}}\int_{\mathbb{R}^d} f(\xi)\,\mathrm{d}\xi\ = T_{(2\pi)^{-\frac{d}{2}}}(f).$$

Thus we have

- *The Fourier transform of derivatives of Dirac function are polynomials*[66]
$$\widehat{\partial^\alpha \delta} = (2\pi)^{-\frac{d}{2}}(i\xi)^\alpha.$$
- *The (inverse) Fourier transform of the constant function* 1 *is Dirac function*
$$\mathcal{F}(1) = \mathcal{F}^{-1}(1) = (2\pi)^{\frac{d}{2}}\delta,$$
and hence for any $\xi_0 \in \mathbb{R}^d$*,* $A > 0$*,*
$$\mathcal{F}(e^{ix\cdot\xi_0}) = \tau_{\xi_0}\mathcal{F}(1) = (2\pi)^{\frac{d}{2}}\delta(\xi-\xi_0) = (2\pi)^{\frac{d}{2}}\delta_{\xi_0},$$
$$\mathcal{F}(A\sin(x\cdot\xi_0)) = (2\pi)^{\frac{d}{2}}\frac{A}{2i}(\delta_{\xi_0}-\delta_{-\xi_0}).$$

Note that

[66] We have the fact that a tempered distribution T with $\mathrm{Supp}\,(T) = \{0\} \Leftrightarrow \mathcal{F}(T)$ is a polynomial. Thus $T \in \mathcal{S}'(\mathbb{R}^d)$ with $\mathrm{Supp}\,(T) = \{0\}$ can only be a linear combination of δ and its derivatives:
$$T = \sum_{|\alpha|\leqslant k} a_\alpha \partial^\alpha \delta. \tag{3.69}$$

- This matches Example 3.2.

  Indeed, for $d = 1$,

$$\mathrm{rect}(x) = 1_{[-\frac{T}{2},\infty)}(x) - 1_{[\frac{T}{2},\infty)}(x) = (\tau_{-\frac{T}{2}}H)(x) - (\tau_{\frac{T}{2}}H)(x),$$

  where $H(x) = 1_{[0,\infty)}$ is the Heaviside function. We derive from the fact (2.46): $H' = \delta$ that

$$\begin{aligned}\mathcal{F}(\mathrm{rect}') &= \mathcal{F}\big(\tau_{-\frac{T}{2}}H' - \tau_{\frac{T}{2}}H'\big) = e^{i\frac{T}{2}\xi}\mathcal{F}(H') - e^{-i\frac{T}{2}\xi}\mathcal{F}(H') \\ &= \big(e^{i\frac{T}{2}\xi} - e^{-i\frac{T}{2}\xi}\big)\mathcal{F}(\delta) = 2i\sin(\frac{T}{2}\xi)\frac{1}{\sqrt{2\pi}},\end{aligned}$$

  which, together with $\mathcal{F}(\mathrm{rect}') = i\xi\mathcal{F}(\mathrm{rect})$, implies (3.66).

- This matches the fact $\mathcal{F}^{-1}\mathcal{F} = \mathrm{Id}$ (recalling the proof given in (3.62)):

$$\begin{aligned}\mathcal{F}^{-1}\mathcal{F}(f)(x) &= (2\pi)^{-d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{-i(y-x)\cdot\xi}\,\mathrm{d}\xi\, f(y)\,\mathrm{d}y \\ &= (2\pi)^{-\frac{d}{2}}\langle\mathcal{F}(1)(\cdot - x), f\rangle_{\mathcal{S}',\mathcal{S}} \\ &= \langle\delta(\cdot - x), f\rangle_{\mathcal{S}',\mathcal{S}} = f(x).\end{aligned}$$

Example 3.5. *If f is homogeneous of degree m: $f(\lambda x) = \lambda^m f(x)$, then $\hat{f}$ is homogeneous of degree $-(m+d)$: $\hat{f}(\lambda^{-1}\xi) = \lambda^{m+d}\hat{f}(\xi)$. For example, for $\sigma \in (0, d)$, then*

$$\mathcal{F}(|x|^{-\sigma}) = c|\xi|^{\sigma-d} \tag{3.70}$$

for some constant c.
We give the ideas here. Let $d \geqslant 2$. Define the operators

$$S = \sum_{j=1}^{d} x_j\partial_{x_j}, \quad Z_{j,k} = x_j\partial_{x_k} - x_k\partial_{x_j},$$

such that

$$|x|^2\partial_{x_k} = x_k S + \sum_{j=1}^{d} x_j Z_{j,k}, \quad S(|x|^{-\sigma}) = -\sigma|x|^{-\sigma}, \quad Z_{j,k}(|x|^{-\sigma}) = 0.$$

Then

$$S(\mathcal{F}(|x|^{-\sigma})) = \sum_{j=1}^{d}\xi_j\partial_{\xi_j}(\mathcal{F}(|x|^{-\sigma})) = \sum_{j=1}^{d}\partial_{\xi_j}(\xi_j\mathcal{F}(|x|^{-\sigma})) - d\mathcal{F}(|x|^{-\sigma})$$

$$= -\mathcal{F}(S(|x|^{-\sigma})) - d\mathcal{F}(|x|^{-\sigma}) = (\sigma - d)\mathcal{F}(|x|^{-\sigma}),$$
$$Z_{j,k}(\mathcal{F}(|x|^{-\sigma})) = -\mathcal{F}((\partial_{x_j}(x_k|x|^{-\sigma}) - \partial_{x_k}(x_j|x|^{-\sigma})) = 0,$$

and thus

$$|\xi|^2\partial_{\xi_k}(|\xi|^{d-\sigma}\mathcal{F}(|x|^{-\sigma})) = \xi_k S(|\xi|^{d-\sigma}\mathcal{F}(|x|^{-\sigma})) + \sum_{j=1}^{d}\xi_j Z_{j,k}(|\xi|^{d-\sigma}\mathcal{F}(|x|^{-\sigma})) = 0.$$

Therefore the distribution $\nabla(|\xi|^{d-\sigma}\mathcal{F}(|x|^{-\sigma}))$ *is supported at* $\{0\}$ *such that* $T := |\xi|^{d-\sigma}\mathcal{F}(|x|^{-\sigma}) - c$ *is supported at* $\{0\}$ *for some constant* c*. By* (3.69)*:* $T = \sum_{|\alpha|\leqslant k} a_\alpha \partial^\alpha\delta$ *and*

$$0 = ST = \sum_{|\alpha|\leqslant k} a_\alpha S\big(\partial^\alpha\delta\big) = -\sum_{|\alpha|\leqslant k} a_\alpha(d + |\alpha|)\partial^\alpha\delta,$$

where we used $S(\partial^\alpha\delta) = -(d + |\alpha|)\partial^\alpha\delta$ *(from the fact* $\partial^\alpha\delta = \widehat{(\frac{1}{i}x)^\alpha(2\pi)^{-\frac{d}{2}}}$*). Hence* $a_\alpha = 0$*,* $T = 0$ *and* $\mathcal{F}(|x|^{-\sigma}) = c|\xi|^{\sigma-d}$*.*
For $d = 1$*, we simply make use of the operator* $S = x\frac{d}{dx}$*.*

3.2.3 Fourier transform on $L^p(\mathbb{R}^d)$

Since any function $f \in L^p(\mathbb{R}^d)$, $p \in [1, \infty]$ can be identified with a tempered distribution $T_f \in \mathcal{S}'(\mathbb{R}^d)$ which is defined as

$$T_f(\phi) = \int_{\mathbb{R}^d} f\phi \,\mathrm{d}x\,, \quad \forall\phi \in \mathcal{S}(\mathbb{R}^d),$$

we can define the Fourier transform of f as the following tempered distribution

$$\mathcal{F}(f)(\phi) = T_f(\mathcal{F}(\phi)), \quad \forall\phi \in \mathcal{S}(\mathbb{R}^d).$$

Furthermore,

- Case $p = 2$: The Fourier transform defines a unitary operator:

  $$\mathcal{F} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d),$$

  and the Fourier transform of $f \in L^2(\mathbb{R}^d)$ is a function in $L^2(\mathbb{R}^d)$ which reads as the following limit (almost everywhere)

  $$\mathcal{F}(f)(\xi) = \lim_{R\to\infty} \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{|x|<R} e^{-ix\cdot\xi} f(x)\,\mathrm{d}x\,, \tag{3.71}$$

  satisfying the Plancherel's identity (3.64): $\|f\|_{L^2(\mathbb{R}^d)} = \|\mathcal{F}(f)\|_{L^2(\mathbb{R}^d)}$.

Indeed, if $f \in L^2$, then $\hat{f} \in L^2$ follows from the density embedding $\mathcal{S}(\mathbb{R}^d) \subset L^2(\mathbb{R}^d)$ and Plancherel's identity (3.64), and the fact that the Fourier transform is unitary follows from Parseval's identity (3.63). Finally, since $f1_{B_R(0)}$ converges to f in L^2-sense, the continuity of the Fourier transform implies the limit (3.71) in L^2-sense. This verifies (3.56).

- Case $p \in (1,2)$: The Fourier transform defines a linear and bounded operator

$$\mathcal{F} : L^p(\mathbb{R}^d) \to L^{p'}(\mathbb{R}^d), \quad p \in [1,2].$$

satisfying

$$\|\mathcal{F}(f)\|_{L^{p'}(\mathbb{R}^d)} \leqslant (2\pi)^{-\frac{d}{2}(\frac{2}{p}-1)} \|f\|_{L^p(\mathbb{R}^d)}. \tag{3.72}$$

Indeed, firstly, we decompose $f \in L^p(\mathbb{R}^d)$ into two parts

$$f = f_1 + f_2 \text{ with } f_1 = f1_{\{x \in \mathbb{R}^d \,||f(x)|>1\}} \text{ and } f_2 = f - f_1,$$

such that $f_1 \in L^1(\mathbb{R}^d)$, $f_2 \in L^2(\mathbb{R}^d)$, and hence $\mathcal{F}(f) = \mathcal{F}(f_1) + \mathcal{F}(f_2) \in L^\infty(\mathbb{R}^d) + L^2(\mathbb{R}^d)$ is a Lebesgue measurable function. Further, since the Fourier transform is a linear operator

$$\begin{aligned}
&\mathcal{F} : L^1(\mathbb{R}^d) \mapsto L^\infty(\mathbb{R}^d) \text{ with } \|\mathcal{F}\|_{L^1(\mathbb{R}^d) \mapsto L^\infty(\mathbb{R}^d)} \leqslant \frac{1}{(2\pi)^{\frac{d}{2}}}, \\
&\mathcal{F} : L^2(\mathbb{R}^d) \mapsto L^2(\mathbb{R}^d) \text{ with } \|\mathcal{F}\|_{L^2(\mathbb{R}^d) \mapsto L^2(\mathbb{R}^d)} = 1,
\end{aligned}$$

(3.72) follows from Riesz-Thorin Interpolation theorem by virtue of

$$\frac{1}{p} = \frac{2(\frac{1}{p}-\frac{1}{2})}{1} + \frac{2(1-\frac{1}{p})}{2}, \quad \frac{1}{p'} = \frac{2(\frac{1}{p}-\frac{1}{2})}{\infty} + \frac{2(1-\frac{1}{p})}{2}.$$

- Case $p > 2$: We don't know whether the Fourier transform of a $L^p(\mathbb{R}^d)$, $p > 2$-function still belongs to some Lebesgue space, and we have seen that the Fourier transform of the constant function $f(x) = 1$ (which belongs to $L^\infty(\mathbb{R}^d)$) is a multiple of the Dirac-function.

[Oct 27, 2026]
[Oct 28, 2026]

4 Preliminaries on Fourier analysis

4.1 Differential equations with constant coefficients

We search for a solution $u = u(x) : \mathbb{R}^d \to \mathbb{C}$ of a differential equation of the form

$$Lu = f, \tag{4.73}$$

where L is a differential operator with constant coefficients

$$L = \sum_{|\alpha| \leqslant m} a_\alpha \partial_x^\alpha, \quad a_\alpha \in \mathbb{C},$$

and $f \in \mathcal{D}'(\mathbb{R}^d)$. If there exists some distribution $E \in \mathcal{D}'(\mathbb{R}^d)$ such that

$$LE = \delta, \tag{4.74}$$

where $\delta = \delta_0$ is the Dirac function, then E is called the fundamental solution of (4.73). The convolution $E * f$, when it is well-defined as a distribution, solves (4.73), since

$$L(E * f) = (LE) * f = \delta * f = f.$$

A natural idea to compute the fundamental solution is to use Fourier transform: We apply Fourier transform to (4.74) to arrive at

$$\sum_{|\alpha| \leqslant m} a_\alpha (i\xi)^\alpha \hat{E}(\xi) = (2\pi)^{-\frac{d}{2}},$$

which leads to, when the polynomial is non vanishing,

$$\hat{E}(\xi) = (2\pi)^{-\frac{d}{2}} \frac{1}{\sum_{|\alpha| \leqslant m} a_\alpha (i\xi)^\alpha},$$

and thus

$$E(x) = (2\pi)^{-\frac{d}{2}} \mathcal{F}^{-1}\Big(\frac{1}{\sum_{|\alpha| \leqslant m} a_\alpha (i\xi)^\alpha}\Big).$$

It seems that the zeros of the polynomial would be problematic, nevertheless, Ehrenpreis[67] and Malgrange[68] proved independently the existence of fundamental solutions in distribution space $\mathcal{D}'$, and Hörmander[69] improved the existence result in tempered distribution space $\mathcal{S}'$.

[67] Leon Ehrenpreis (American, 1930-2010), Solution of some problems of division. I. Division by a polynomial of derivation. Amer. J. Math., 1954, 76: 883-903.

[68] Bernard Malgrange (French, 1928-2024), Existence et approximation des solutions des équations aux dérivées partielles et des équations de convolution. Ann. Inst. Fourier (Grenoble), 1956, 6: 271-355.

[69] Lars Valter Hörmander (Swedish, 1931-2012), On the division of distributions by polynomials. Ark. Mat., 1958, 3: 555-568.

It is usually not an easy job to compute explicitly the (inverse) Fourier transform of some given function. We present some typical examples below, where the solutions are understood in the distribution sense.

4.1.1 The ODE example revisited

We search for solutions of the ODE (1.9) on the whole line $\mathbb{R}$:

$$-u''(x) + u(x) = f(x), \quad x \in \mathbb{R}\,. \tag{4.75}$$

Method 1. ODE theory. The fundamental solutions[70] of the homogeneous ODE $-u''(x) + u(x) = 0$ are

$$\phi(x) = e^x, \quad \psi(x) = e^{-x},$$

where $\phi(x) \to 0$ as $x \to -\infty$ and $\psi(x) \to 0$ as $x \to \infty$. Their Wronskian is

$$W(\phi, \psi) = \phi\psi' - \phi'\psi = -2.$$

The Green function, as tempered distribution $\mathcal{S}'(\mathbb{R})$ (with y viewed as parameter), is

$$G(x,y) = \begin{cases} -\frac{\phi(x)\psi(y)}{W(\phi,\psi)} & \text{for } x < y \\ -\frac{\phi(y)\psi(x)}{W(\phi,\psi)} & \text{for } x > y \end{cases} = \frac{1}{2}e^{-|x-y|}.$$

Thus the solution of (4.75) is the convolution of the Green function and the source term

$$u(x) = \frac{1}{2}\int_{\mathbb{R}} e^{-|x-y|} f(y)\,\mathrm{d}y\,. \tag{4.76}$$

Method 2. Fourier transform. Applying Fourier transform to

$$-E'' + E = \delta, \quad x \in \mathbb{R}$$

yields

$$(\xi^2 + 1)\hat{E}(\xi) = \frac{1}{\sqrt{2\pi}}, \quad \xi \in \mathbb{R}\,.$$

[70] The fundamental solutions are for ODEs, which are different from the definition (4.74) above.

Recall (3.65) that the Fourier transform of the function $\sqrt{\frac{\pi}{2}}e^{-|x|}$ is $\frac{1}{1+\xi^2}$. Thus the fundamental solution of (4.75) is

$$E(x) = \frac{1}{\sqrt{2\pi}}\mathcal{F}^{-1}\big(\frac{1}{1+\xi^2}\big) = \frac{1}{2}e^{-|x|}.$$

Hence the solution is the convolution

$$u(x) = (E * f)(x) = \frac{1}{2}e^{-|\cdot|} * f,$$

which coincides with (4.76).
To conclude, we can formulate the solution of (4.75) in a compact way that

$$u(x) = (1 - \partial_{xx})^{-1} f,$$

where $(1-\partial_{xx})^{-1} = \big(\frac{1}{2}e^{-|\cdot|}\big)*$ denotes the resolvent of the free Schrödinger operator

$$H_0 = -\frac{d^2}{dx^2} : H^2(\mathbb{R}) \to L^2(\mathbb{R})$$

at -1. Note also that

$$G(x,y) = \tau_y E(x) = \tau_x E(y),$$

and hence

$$\begin{aligned}
&(1-\partial_{xx})G(\cdot,y) = \tau_y(1-\partial_{xx})E(x) = \tau_y\delta(x) = \delta(x-y),\\
&\mathcal{F}_{x\mapsto\xi}G(\cdot,y) = \frac{1}{\sqrt{2\pi}}\frac{e^{-iy\xi}}{1+\xi^2}.
\end{aligned}$$

4.1.2 Cauchy-Riemann operator

Let $z = x + iy \in \mathbb{C}$, such that

$$\bar{z} = x - iy, \quad x = \frac{1}{2}(z+\bar{z}), \quad y = \frac{1}{2i}(z-\bar{z}),$$

and hence

$$\partial_z = \frac{1}{2}(\partial_x - i\partial_y), \quad \partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y). \tag{4.77}$$

Define the Cauchy-Riemann operator as $\partial_{\bar{z}}$, and we search for the fundamental solution of the Cauchy-Riemann operator[71]

$$\frac{1}{2}(\partial_x + i\partial_y)E = \delta, \quad (x,y) \in \mathbb{R}^2\,.$$

[71]The Cauchy-Riemann equations for some complex-valued function $F(z) = u(x,y) + iv(x,y)$, $u, v \in \mathbb{R}$ refers to $\begin{cases} \partial_x u = \partial_y v, \\ \partial_y u = -\partial_x v, \end{cases}$, which is equivalent to $\partial_{\bar{z}} F = 0$.

Applying Fourier transform with respect to y-variable yields the following equation for $\hat{E}(x,\eta) = \mathcal{F}_y(E(x,y))(x,\eta)$:

$$\frac{1}{2}\partial_x \hat{E}(x,\eta) - \frac{1}{2}\eta \hat{E}(x,\eta) = \frac{1}{\sqrt{2\pi}}\delta(x) \otimes 1(\eta).$$

Therefore

$$\frac{1}{2}\partial_x(e^{-x\eta}\hat{E}) = \frac{1}{\sqrt{2\pi}}e^{-x\eta}\delta(x) \otimes 1(\eta) = \frac{1}{\sqrt{2\pi}}\delta(x) \otimes 1(\eta),$$

and we derive from (2.46): $\partial_x H(x) = \delta(x)$ that

$$\frac{1}{2}e^{-x\eta}\hat{E}(x,\eta) = \frac{1}{\sqrt{2\pi}}(H(x) + C(\eta)),$$

that is,

$$\hat{E}(x,\eta) = \frac{2}{\sqrt{2\pi}}(H(x) + C(\eta))e^{x\eta} = \begin{cases} \frac{2}{\sqrt{2\pi}}C(\eta)e^{x\eta}, & x < 0 \\ \frac{2}{\sqrt{2\pi}}(1 + C(\eta))e^{x\eta}, & x \geqslant 0 \end{cases}.$$

The restriction that $\hat{E} \in \mathcal{S}'(\mathbb{R}^2)$ requires that

$$C(\eta) = \begin{cases} 0, & \eta < 0 \\ -1, & \eta \geqslant 0 \end{cases},$$

and hence

$$\hat{E}(x,\eta) = \begin{cases} \frac{2}{\sqrt{2\pi}}H(x)e^{x\eta}, & \eta < 0 \\ \frac{2}{\sqrt{2\pi}}(H(x) - 1)e^{x\eta}, & \eta \geqslant 0 \end{cases}.$$

We compute its inverse Fourier transform with respect to the variable η to arrive at the explicit formulation of the fundamental solution

$$\begin{aligned} E(x,y) &= \frac{2}{2\pi}\Big(\int_{-\infty}^{0} e^{iy\eta}H(x)e^{x\eta}\,d\eta + \int_{0}^{\infty} e^{iy\eta}(H(x) - 1)e^{x\eta}\,d\eta\Big) \\ &= \frac{1}{\pi}\Big(H(x)\frac{1}{x+iy} - (H(x) - 1)\frac{1}{x+iy}\Big) = \frac{1}{\pi}\frac{1}{x+iy} = \frac{1}{\pi}\frac{1}{z}. \end{aligned} \quad (4.78)$$

Thus the function

$$u = E * f = \frac{1}{\pi}\int_{\mathbb{C}} \frac{1}{z-w}f(w)dw,$$

when it is well-defined (e.g. if $f \in \mathcal{D}(\mathbb{C})$), solves the inhomogeneous Cauchy-Riemann equation

$$\partial_{\bar{z}} u = f \text{ on } \mathbb{C}\,.$$

Note that adding any holomorphic function to u yields another solution. Hence, the solution is not unique. To obtain uniqueness, one typically imposes suitable additional conditions, such as appropriate boundary conditions or growth conditions at infinity.

Boundary value of E. Formally, $\lim_{y\to 0} E(x,y) = \lim_{y\to 0} \frac{1}{\pi}\frac{1}{x+iy} = \frac{1}{\pi}\frac{1}{x}$, but is it true? For any fixed $x \neq 0$ such that the denominator never vanishes for all $y \in \mathbb{R}$, this pointwise convergence is valid. However, we can not simply take the limit in the expression of E, as it is not defined pointwisely, and in particular not at $(0,0)$. It is locally integrable and hence a tempered distribution in $\mathcal{S}'(\mathbb{R}^2)$.
We compute the limit of E as $y \to 0_\pm$ in the sense of distribution. Firstly, for any fixed $y \neq 0$, $E(\cdot, y) \in L^1_{\text{loc}}(\mathbb{R}) \subset \mathcal{S}'(\mathbb{R})$. Then since

$$\frac{1}{x+iy} = \frac{x-iy}{x^2+y^2} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2},$$

we apply the tempered distribution $E(\cdot, y)$ (with y viewed as a parameter) to a real-valued Schwartz function $\varphi(x)$ to arrive at

$$\begin{aligned}
\langle E(\cdot,y), \varphi\rangle_{\mathcal{S}',\mathcal{S}} &= \frac{1}{\pi}\int_{\mathbb{R}} \frac{x}{x^2+y^2}\varphi(x)\,\mathrm{d}x \; - i\frac{1}{\pi}\int_{\mathbb{R}} \frac{y}{x^2+y^2}\varphi(x)\,\mathrm{d}x \\
&= \frac{1}{\pi}\int_0^\infty \frac{x}{x^2+y^2}\big(\varphi(x)-\varphi(-x)\big)\,\mathrm{d}x \; - i\frac{1}{\pi}\mathrm{sgn}\,(y)\int_{\mathbb{R}} \frac{y^2}{y^2(t^2+1)}\varphi(yt)\,\mathrm{d}t \\
&\overset{y\to 0_\pm}{\to} \Big\langle \text{p.v.}\,(\frac{1}{\pi}\frac{1}{x}), \varphi \Big\rangle_{\mathcal{S}',\mathcal{S}} \mp i\varphi(0).
\end{aligned}$$

by use of the definition (3.68) of the tempered distribution p.v. $(\frac{1}{\pi}\frac{1}{x})$. Therefore we arrive at the Plemelj formula

$$\lim_{y\to 0_\pm} E(x,y) = \text{p.v.}\,(\frac{1}{\pi}\frac{1}{x}) \mp i\delta(x), \tag{4.79}$$

which holds in the sense of distribution.

[Oct 28, 2026]
[Nov 3, 2026]

4.1.3 Laplace equation on the upper half plane

We are going to solve the Dirichlet problem of the Laplace equation on the upper half plane:

$$\begin{cases} \Delta u = 0 & \text{if } x \in \mathbb{R},\, y \in (0,\infty), \\ u = f & \text{if } x \in \mathbb{R},\, y = 0, \end{cases} \tag{4.80}$$

by use of the fundamental solution of the Cauchy-Riemann operator given above.

Hilbert transform, w.r.t. variable $x \in \mathbb{R}$. Recall the definition of conjugate function (1.28), and we can define the counter version on the whole line $\mathbb{R}$:

$$\mathcal{H}(f)(x) = \mathcal{F}^{-1}\big((\frac{1}{i}\mathrm{sgn}\,(\xi))\hat{f}(\xi)\big). \tag{4.81}$$

Recall (3.67): $\mathcal{F}(\text{p.v.}\,\frac{1}{\pi}\frac{1}{x}) = \frac{\frac{1}{i}\mathrm{sgn}\,(\xi)}{\sqrt{2\pi}}$, and therefore if $f \in \mathcal{S}(\mathbb{R}^d)$,

$$\mathcal{H}(f)(x) = \text{p.v.}\,(\frac{1}{\pi}\frac{1}{\cdot}) * f = \text{p.v.}\,\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(s)}{x-s}\,\mathrm{d}s\,, \tag{4.82}$$

which is understood as (3.68):

$$\mathcal{H}(f)(x) = \frac{1}{\pi}\int_0^{\infty}\frac{f(x-s)-f(x+s)}{s}\,\mathrm{d}s\,.$$

This is called Hilbert transform[72] $\mathcal{H} : f \mapsto \mathcal{H}(f)$. It is a Fourier multiplier operator with the symbol

$$m(\xi) = \frac{1}{i}\mathrm{sgn}\,(\xi),$$

which satisfies

- Isometry on $L^2(\mathbb{R})$: By Plancheral's identity,

$$\|\mathcal{H}(f)\|_{L^2(\mathbb{R})}^2 = \|\widehat{\mathcal{H}(f)}\|_{L^2(\mathbb{R})}^2 = \|\hat{f}\|_{L^2(\mathbb{R})}^2 = \|f\|_{L^2(\mathbb{R})}^2.$$

- Unitary and involution on $L^2(\mathbb{R})$:

$$\mathcal{H}^2 = \mathcal{H}\mathcal{H} = -\mathrm{Id}\,, \quad \mathcal{H}^* = \mathcal{H}^{-1} = -\mathcal{H}.$$

Similar as the L^p-boundedness of the conjugate function of $f \in L^p([-\pi,\pi])$, $p \in (1,\infty)$, the Hilbert transform is also bounded in $L^p(\mathbb{R})$, $p \in (1,\infty)$. It can follow from the celebrated Mihlin-Hörmander multiplier theorem[73]

[72] David Hilbert (German, 1862-1943), Über eine Anwendung der Integralgleichungen auf ein Problem der Funktionentheorie. *Verhandlungen des Dritten Internationalen Mathematiker-Kongresses in Heidelberg* vom 8. bis 13. August 1904, edited by A. Krazer, B. G. Teubner, Leipzig, 1905.

Godfrey Harold Hardy (British, 1877-1947), Notes on some points in the integral calculus LVIII: On Hilbert transforms. *Messenger of Mathematics*, vol. 54, pp. 20–27, 1925.

[73] Solomon G. Mihlin, On the multipliers of Fourier integrals. *Doklady Akademii Nauk SSSR* 109, 701-703, 1956.

Lars Hörmander, Estimates for translation invariant operators in L^p spaces. *Acta Mathematica* 104, 93-140, 1960.

regarding the $L^p(\mathbb{R})$ boundedness of Fourier multiplier operators associated with sufficiently decaying symbols: Let $m(\xi)$ be a complex-valued bounded function on $\mathbb{R}\backslash\{0\}$ such that

$$M := \|m\|_{L^\infty} + \sup_{R>0}\Big(\frac{1}{R}\int_{R<|\xi|<2R}\big(|m(\xi)|^2 + |Rm'(\xi)|^2\big)\,\mathrm{d}\xi\Big)^{\frac{1}{2}} < \infty. \quad (4.83)$$

Then the associated Fourier multiplier operator $\mathcal{M} : f \mapsto \mathcal{F}^{-1}(m(\xi)\hat{f}(\xi))$ is bounded on $L^p(\mathbb{R})$, $p \in (1,\infty)$, with bounded operator norm as

$$\|\mathcal{M}\|_{L^p(\mathbb{R})\to L^p(\mathbb{R})} \leqslant C\max\{p, \frac{1}{p-1}\}M.$$

The symbol $m(\xi) = -i\mathrm{sgn}\,(\xi)$ of the Hilbert transform satisfies (4.83).

Cauchy integral, w.r.t. variable $z \in \mathbb{C}^+$ or $\mathbb{C}^-$. Let $z \in \mathbb{C}^+$ be a complex number in the upper half complex plane, and let $f = f(s) : \mathbb{R} \to \mathbb{C}$ be defined on the real line. Then the Cauchy-type integral on the real line reads

$$F_+(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{f(s)}{s-z}\,\mathrm{d}s\,.$$

If f is a regular (say, Hölder continuous) function, then $F_+(z)$ is a holomorphic function on $\mathbb{C}^+$. Similarly we can define a holomorphic function $F_-(z)$ on the lower half complex plane $\mathbb{C}^-$.
Recall (4.78): $E(z) = \frac{1}{\pi}\frac{1}{z}$, and for any $y \neq 0$, $E(\cdot, y) \in L^\infty(\mathbb{R}) \subset \mathcal{S}'(\mathbb{R})$. Then if $f \in \mathcal{S}(\mathbb{R})$,

$$\begin{aligned} F_+(z) &= \frac{i}{2}E(\cdot,y) *_x f = \frac{i}{2}\frac{1}{\pi}\int_{\mathbb{R}}\frac{f(s)}{(x-s)+iy}\,\mathrm{d}s\,, \qquad \text{if } y>0 \Leftrightarrow z\in\mathbb{C}^+,\\ F_-(z) &= \frac{i}{2}E(\cdot,y) *_x f = \frac{i}{2}\frac{1}{\pi}\int_{\mathbb{R}}\frac{f(s)}{(x-s)+iy}\,\mathrm{d}s\,, \qquad \text{if } y<0 \Leftrightarrow z\in\mathbb{C}^-\,. \end{aligned}$$

The Plemelj identity (4.79): $\lim_{y\to 0_\pm} E(x,y) = \text{p.v.}\,(\frac{1}{\pi}\frac{1}{x}) \mp i\delta(x)$, and the definition of Hilbert transform (4.82): $\mathcal{H} = (\text{p.v.}\,\frac{1}{\pi}\frac{1}{x})*$ lead to the following behaviours of $F_\pm(z)$ approaching the real line:

$$\begin{aligned} \lim_{y\to 0_+} F_+(z) &= \frac{i}{2}\lim_{y\to 0_+}\langle E(x-\cdot,y), f\rangle_{\mathcal{S}',\mathcal{S}} = \frac{1}{2}f(x) + \frac{i}{2}\mathcal{H}(f)(x),\\ \lim_{y\to 0_-} F_-(z) &= \frac{i}{2}\lim_{y\to 0_-}\langle E(x-\cdot,y), f\rangle_{\mathcal{S}',\mathcal{S}} = -\frac{1}{2}f(x) + \frac{i}{2}\mathcal{H}(f)(x). \end{aligned} \quad (4.84)$$

The limits are NOT the formal limits by simply replacing z by x:

$$\text{p.v.}\,\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{f(s)}{s-x}\,\mathrm{d}s\;=\frac{i}{2}\mathcal{H}(f)(x),$$

but rather with additional correction terms $\pm\frac{1}{2}f(x)$, which correspond to half the residue of $\frac{f(s)}{s-z}$ at the point $(x,0)$, anticlockwisely resp. clockwisely. The jump of the two limits on the real line is the function itself

$$\lim_{y\to 0_+}F_+(z)-\lim_{y\to 0_-}F_-(z)=f(x),$$

while their average is the Hilbert transform (up to i):

$$\lim_{y\to 0_+}F_+(z)+\lim_{y\to 0_-}F_-(z)=i\mathcal{H}(f)(x).$$

The functions $F_\pm(z)$ are the solution of the classical Riemann-Hilbert problem: Given a regular (smooth and decaying) function $f(x)$ on the real line, find a function $F(z)$ such that

- Holomorphy: $F(z)$ is holomorphic on $\mathbb{C}^+$ and $\mathbb{C}^-$;
- Decaying property: $F(z)\to 0$ as $|z|\to\infty$;
- Jump condition: The boundary values from above F_+ and from below F_- on the real axis satisfy the jump condition

$$F_+(x)-F_-(x)=f(x),\quad x\in\mathbb{R}\,.$$

The Fourier transform of the Plemelj identity (4.84) is

$$\begin{aligned}\mathcal{F}\Big(\lim_{y\to 0_+}F_+(z)\Big)(\xi)&=\frac{1}{2}\hat{f}(\xi)+\frac{i}{2}(-i\mathrm{sgn}\,(\xi))\hat{f}(\xi)=1_{\xi>0}\hat{f}(\xi),\\ \mathcal{F}\Big(\lim_{y\to 0_-}F_-(z)\Big)(\xi)&=-\frac{1}{2}\hat{f}(\xi)+\frac{i}{2}(-i\mathrm{sgn}\,(\xi))\hat{f}(\xi)=-1_{\xi<0}\hat{f}(\xi),\end{aligned}$$

which resembles the Fourier transform of the Riesz projections $P_\pm(f)$ in (1.29), (1.30): The Riesz projection $P_\pm(f)$, as the boundary value of $F_\pm(z)$ on $\mathbb{R}$, filters the positive resp. negative frequencies of the function f. The Hilbert transform, with symbol $-i\mathrm{sgn}\,(\xi)$, provides a $\frac{\pi}{2}$ degree phase shift at all negative frequencies, and a $-\frac{\pi}{2}$ degree phase shift at all positive frequencies.

Poisson kernel, w.r.t. variable $x \in \mathbb{R}$ and parameter $y \in \mathbb{R}^+$. Let us revisit the limits (4.84) in terms of Poisson kernel and its conjugate.
Let f be a real-valued function, $z = x + iy$ and we separate the real and imaginary parts of the function $F_+(z)$:

$$F_+(z) = \frac{i}{2} E(\cdot, y) * f = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{y}{(x-s)^2 + y^2} f(s)\, \mathrm{d}s \\ + i \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{x-s}{(x-s)^2 + y^2} f(s)\, \mathrm{d}s \ =: \frac{1}{2}(P_y * f)(x) + \frac{i}{2}(Q_y * f)(x),$$

where $P_y(\cdot)$ denotes the Poisson kernel

$$P_y(x) = \frac{1}{\pi} \frac{y}{x^2 + y^2} = -\mathrm{Im}\,[E], \tag{4.85}$$

and the conjugate Poisson kernel reads

$$Q_y(x) = \frac{1}{\pi} \frac{x}{x^2 + y^2} = \mathrm{Re}\,[E]. \tag{4.86}$$

The Plemelj identities (4.84) imply then

$$P_y * f \to f, \text{ as } y \to 0_+, \\ Q_y * f \to \mathcal{H}(f), \text{ as } y \to 0_+.$$

The convergence is rigorous in $L^p(\mathbb{R})$, $p \in [1, \infty)$: For any $f \in L^p(\mathbb{R})$, $p \in [1, \infty)$,[74]

$$\|P_y * f - f\|_{L^p(\mathbb{R})} \overset{y \to 0_+}{\to} 0, \quad \|Q_y * f - \mathcal{H}(f)\|_{L^p(\mathbb{R})} \overset{y \to 0_+}{\to} 0.$$

Indeed, similar as the Poisson kernel $(P_r)_{r \in [0,1)}$ given by (1.33) on the disk, the Poisson kernel $(P_y)_{y>0}$ on the upper half plane is an approximate identity satisfying (1.34): For $y > 0$,

$$P_y(x) = \frac{1}{y} P_1(\frac{x}{y}),$$

and thus

$$\int_{\mathbb{R}} P_y\, \mathrm{d}x \ = \|P_y\|_{L^1(\mathbb{R})} = \|P_1\|_{L^1(\mathbb{R})} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{x^2 + 1}\, \mathrm{d}x \ = \frac{1}{\pi} \arctan(x)|_{-\infty}^{\infty} = 1.$$

[74] The endpoint case $p = \infty$ is excluded.

Furthermore, for any fixed $\delta > 0$,

$$\int_{|x|>\delta} P_y(x)\,\mathrm{d}x = \frac{1}{\pi}\Big(\arctan(x)|_{\frac{\delta}{y}}^{\infty} + \arctan(x)|_{-\infty}^{-\frac{\delta}{y}}\Big) \to 0 \text{ as } \delta \to 0_+.$$

To conclude, the convolution

$$u(x,y) = P_y * f$$

solves (4.80) in the sense as for (1.35) in Subsection 1.4.2.

4.1.4 Poisson equation

For given source function $f = f(x) \in \mathcal{D}'(\mathbb{R}^d)$, we aim to solve the Poisson equation

$$\Delta u = f \text{ on } \mathbb{R}^d. \tag{4.87}$$

Case $d = 2$ In terms of the complex variable $z = x + iy$ on $\mathbb{R}^2$, the Laplace operator becomes (recalling (4.77))

$$\Delta = \partial_{xx} + \partial_{yy} = (\partial_x - i\partial_y)(\partial_x + i\partial_y) = 4\partial_z\partial_{\bar{z}} =: 4\partial\bar{\partial}.$$

Recall the fundamental solution (4.78):

$$\bar{\partial}E_{\bar{\partial}} = \bar{\partial}(\frac{1}{\pi}\frac{1}{z}) = \delta(z).$$

Since

$$\partial(\frac{1}{2\pi}\log|z|) = \frac{1}{4\pi}\frac{1}{z} = \frac{1}{4}E_{\bar{\partial}},$$

we have

$$\Delta(\frac{1}{2\pi}\log|z|) = 4\bar{\partial}\big(\frac{1}{4}E_{\bar{\partial}}\big) = \delta(z).$$

That is, the Laplace equation $\Delta E = \delta$ has the fundamental solution

$$E_\Delta(x,y) = \frac{1}{2\pi}\log\sqrt{x^2+y^2}, \quad x,y \in \mathbb{R}, \tag{4.88}$$

and the convolution

$$u(x,y) = E_\Delta * f$$

solves the Poisson equation (4.87) on $\mathbb{R}^2$. Notice that the fundamental solution $E_{\bar\partial}$ of the Cauchy-Riemann equation is four times the derivative of E_Δ:

$$E_{\bar\partial} = 4\partial_z E_\Delta,$$

the Poisson kernel is twice the normal derivative of E_Δ:

$$P_y(x) = -\mathrm{Im}\,[E_{\bar\partial}] = -\mathrm{Im}\,[4\partial_z E_\Delta] = 2\partial_y E_\Delta,$$

while the conjugate Poisson kernel is twice the tangential derivative of E_Δ:

$$Q_y(x) = \mathrm{Re}\,[E_{\bar\partial}] = \mathrm{Re}\,[4\partial_z E_\Delta] = 2\partial_x E_\Delta.$$

Case $d \geqslant 3$ The fundamental solution $E = E_\Delta$ of the Poisson equation (4.87) satisfies $\Delta E = \delta$, and thus

$$-|\xi|^2 \hat{E}(\xi) = (2\pi)^{-\frac{d}{2}}.$$

By virtue of (3.70): $\mathcal{F}(|x|^{-\sigma}) = c|\xi|^{\sigma-d}$, $\sigma \in (0,d)$, for some constant $c \in \mathbb{R}$, we derive

$$E(x) = c_d |x|^{-(d-2)}, \text{ for some } c_d \in \mathbb{R}\,.$$

We now compute c_d: Since for any $\varphi \in \mathcal{S}$,

$$\langle E, \varphi\rangle_{\mathcal{S}',\mathcal{S}} = \langle \hat{E}, \mathcal{F}^{-1}\varphi\rangle_{\mathcal{S}',\mathcal{S}},$$

we take $\varphi = e^{-\frac{1}{2}|\cdot|^2}$ the Gaussian function to derive

$$c_d \int_{\mathbb{R}^d} |x|^{2-d} e^{-\frac{1}{2}|x|^2}\,\mathrm{d}x \;= -(2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} \frac{1}{|\xi|^2} e^{-\frac{1}{2}|\xi|^2}\,\mathrm{d}\xi\,.$$

Let ω_d denote the area of the unit sphere $\S^d$, then the left hand side is

$$c_d\omega_d \int_0^\infty r e^{-\frac{1}{2}r^2}\,\mathrm{d}r \;= c_d\omega_d,$$

and the right hand side is

$$-(2\pi)^{-\frac{d}{2}}\omega_d \int_0^\infty r^{d-3} e^{-\frac{1}{2}r^2}\,\mathrm{d}r\,.$$

We conclude by integration by parts and induction argument that

$$c_d = -(2\pi)^{-\frac{d}{2}} \int_0^\infty r^{d-3} e^{-\frac{1}{2}r^2}\,\mathrm{d}r \;= -\frac{1}{(d-2)\omega_d}.$$

Finally the convolution $u = E * f = c_d|\cdot|^{2-d} * f$ solves the Poisson equation (4.87) on R^d, $d \geqslant 3$.

4.1.5 Heat equation

Consider the initial value problem for the heat equation

$$\partial_t u - \Delta u = 0 \text{ in } (0,\infty)\times\mathbb{R}^d, \quad u|_{t=0} = f(x) \text{ in } \mathbb{R}^d\,. \tag{4.89}$$

We search for the fundamental solution in $\mathcal{S}'(\mathbb{R}^{d+1})$ satisfying

$$\partial_t E - \Delta E = \delta(t,x), \quad E|_{t=0} = \delta(x), \quad (t,x)\in\mathbb{R}^{d+1}\,.$$

Applying Fourier transform with respect to the x-variable yields

$$\partial_t \hat{E} + |\xi|^2\hat{E} = (2\pi)^{-\frac{d}{2}}\delta(t)\otimes 1(\xi), \quad \hat{E}|_{t=0} = (2\pi)^{-\frac{d}{2}}, \quad (t,\xi)\in\mathbb{R}^{d+1}\,.$$

Thus

$$\begin{aligned}
&\partial_t(e^{t|\xi|^2}\hat{E}) = (2\pi)^{-\frac{d}{2}}(\partial_t H(t))\otimes 1(\xi), \quad (e^{t|\xi|^2}\hat{E})|_{t=0} = (2\pi)^{-\frac{d}{2}}, \quad (t,\xi)\in\mathbb{R}^{d+1}\\
&\Longrightarrow e^{t|\xi|^2}\hat{E} = (2\pi)^{-\frac{d}{2}}H(t), \quad (t,\xi)\in\mathbb{R}^{d+1}\\
&\Longrightarrow \hat{E} = (2\pi)^{-\frac{d}{2}}H(t)e^{-t|\xi|^2}, \quad (t,\xi)\in\mathbb{R}^{d+1}\,.
\end{aligned}$$

Therefore

$$E(t,x) = \begin{cases} (2\pi)^{-\frac{d}{2}}\mathcal{F}^{-1}_{\xi\to x}\big(e^{-\frac{1}{2}(2t)|\xi|^2}\big) = \frac{e^{-\frac{|x|^2}{4t}}}{(4\pi t)^{\frac{d}{2}}}, & t>0,\\ \delta(x), & t=0,\\ 0, & t<0. \end{cases}$$

To conclude, the fundamental solution $E_t(x) = E(t,x)$ is a Dirac sequence parametrized by $t>0$:

$$E_t(x) = \frac{e^{-\frac{|x|^2}{4t}}}{(4\pi t)^{\frac{d}{2}}} \xrightarrow{t\to 0_+} \delta(x) \text{ in } \mathcal{S}'(\mathbb{R}^d),$$

and the convolution

$$u(t,x) = E(t,\cdot)*f = \frac{1}{(4\pi t)^{\frac{d}{2}}}\int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{4t}} f(y)\,\mathrm{d}y\,, \quad (t,x)\in(0,\infty)\times\mathbb{R}^d$$

satisfies the initial value problem (4.89) in the usual sense.

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