① N-body problem (C.M.)

A famous topic from C.M., interesting for both mathematician, physicist, astronomist, engineers.

When talking about N-body, we have lots of pictures/problems in our mind.

~~E.g.~~ For instance, 2-body problem

3-body problem

Solar system has 8 planets.

We may also have some names in our mind:

Newton, Kepler, Poincarè

We deal with N-body mathematically/physically:

N point masses in a inertial reference system $\mathbb{R}^3$

They follow Newton's gravitational law.

For i-th, j-th bodies $\quad F \propto \dfrac{m_i m_j}{r_{ij}}$

force between m_i, m_j

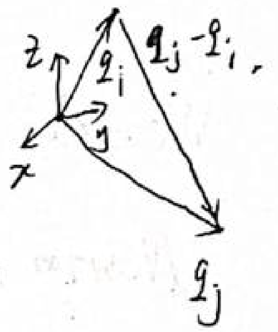
particles
point masses: $m_1, \cdots, m_N$, $m_i > 0$.

positions vectors $(q_1, \cdots, q_N) \in \mathbb{R}^{3N}$

According to Newton's law, the equations of motion are

$$(*) \quad m_i \ddot{q}_i = \sum_{j=1, j\neq i}^{N} \frac{G m_i m_j (q_j - q_i)}{\| q_i - q_j \|^3} \quad \forall i = 1, \cdots, N$$

$$= -\frac{\partial V}{\partial q_i}$$



where G is gravitational constant
$\approx 6.67408 \times 10^{-11} \, m^3/sec^2 kg$

$$V = -\sum_{1 \le i < j \le N} \frac{G m_i m_j}{\| q_i - q_j \|} \quad \text{potential}$$

$U = -V$ is called self-potential

②

$$\Delta := \{(q_1, \cdots, q_N) \in \mathbb{R}^{3N} \mid q_i = q_j \text{ for } i \neq j\} \subset \mathbb{R}^{3N}$$

$\uparrow$

collision set

Eqn $(*)$ is well-defined in $\mathbb{R}^{3N} \setminus \Delta$.

Let $q := (q_1, \cdots, q_N) \in \mathbb{R}^{3N}$

$$M := \mathrm{diag}(m_1, m_1, m_1, \cdots, m_N, m_N, m_N)$$

$3N \times 3N$ diagonal matrix

$(*)$ is equivalent to

$$M\ddot{q} - \frac{\partial U}{\partial q} = 0 .$$

Now let's introduce momentum

$$p = (p_1, \cdots, p_N) \in \mathbb{R}^{3N}$$

s.t. $p = M\dot{q}$ namely, $p_i = m_i \dot{q}_i$

Then the eqns of motion are

$$\dot{q}_i = \frac{p_i}{m_i} = \frac{\partial H}{\partial p_i}$$

$$\dot{p}_i = \sum_{\substack{j=1, \\ j \neq i}}^{N} \frac{G m_i m_j (q_j - q_i)}{\| q_i - q_j \|^3} = -\frac{\partial H}{\partial q_i}$$

Where the Hamiltonian is

$$H = \cancel{\tfrac{1}{2}} \sum_{i=1}^{N} \frac{\|p_i\|^2}{2m_i} - \sum_{1 \le i < j \le N} \frac{G m_i m_j}{\|q_i - q_j\|} = T - U$$
$$= T + V$$

and $T := \sum_{i=1}^{N} \frac{\|p_i\|^2}{2m_i} = \frac{1}{2} p^T M^{-1} p$

$$= \sum_{i=1}^{N} \frac{m_i \|\dot{q}_i\|^2}{2} \quad \text{kinetic energy}$$

physical meaning of H : total energy
$\qquad\qquad\qquad\qquad\qquad\qquad ||$
kinectic energy + potential

H is a first integral (constant of motion)

Indeed,

$$\frac{\partial H}{\partial t} = \frac{\partial H}{\partial q} \dot{q} + \frac{\partial H}{\partial p} \dot{p}$$

$$= \frac{\partial H}{\partial q} \frac{\partial H}{\partial p} + \frac{\partial H}{\partial p} \left(-\frac{\partial H}{\partial q}\right)$$

$$= 0$$

H = const (along the ~~m~~ solution, i.e. the motion of given initial condition)

(③

define the center of mass as

$$C := m_1 q_1 + m_2 q_2 + \cdots + m_N q_N$$

(or equivalently $C = \dfrac{\sum_{i=1}^{N} m_i q_i}{\sum_{i=1}^{N} m_i}$)

and total linear momentum

$$L := p_1 + \cdots + p_N$$

obviously,

$$\dot{C} = L$$

and $\dot{L} = 0$.

Indeed,

$$\dot{L} = \sum_{i=1}^{N} \dot{p}_i = \sum_{i=1}^{N} -\frac{\partial H}{\partial q_i}$$

$$= \sum_{i=1}^{N} -\frac{\partial V}{\partial q_i}$$

$$= \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{G m_i m_j (q_j - q_i)}{\| q_i - q_j \|^3}$$

$$= 0$$

Therefore, with given initial conditions q_0, p_0

$$q(0) = q_0, \quad p(0) = p_0$$

we have

$$L = L_0$$

$$C = L_0 t + C_0$$

where L_0, C_0 are constants related to initial conditions.

Let

$$A = \sum_{i=1}^{N} q_i \times m\dot{q}_i = \sum_{i=1}^{N} q_i \times p_i$$

be the total angular momentum. Since

$$\frac{dA}{dt} = \sum_{i=1}^{N} \big(\underbrace{\dot{q}_i \times p_i}_{=0} + q_i \times \dot{p}_i \big)$$

$$= \sum_{i=1}^{N} q_i \times \sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{G m_i m_j (q_j - q_i)}{\| q_j - q_i \|^3}$$

$$= \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{G m_i m_j \, q_i \times q_j}{\| q_j - q_i \|^3}$$

$$= 0$$

So the angular momentum A is a first integral.

In conclusion, we now obtain

H : total energy "1"

C : center of mass "3"

L : total linear momentum "3"

A : total angular momentum "3"

10 integrals of the N-body problem for a system of $6N$ first-order eqns

The spatial N-body problem has the 10 integrals:

3 for linear momentum, 3 for center of mass,

3 for angular momentum, 1 for energy.

The planar N-body problem has the 6 integrals:

2 for linear momentum, 2 for center of mass

1 for angular momentum, 1 for energy

~~Spatial case : phase space is $\mathbb{R}^{12} \setminus \Delta$~~

~~planar case : phase space is $\mathbb{R}^{8} \setminus \Delta$~~

Spatial 2-body problem

phase space is $\mathbb{R}^{12} \setminus \Delta$ (collision set), 10 integrals

planar 2-body problem

phase space is $\mathbb{R}^{8} \setminus \Delta$ (collision set), 6 integrals

Indeed, 2-body problem has an extra integral

Therefore, 2-body problem is solvable.

- Equilibrium and Central Configurations

To have the equilibrium, we require

$$\frac{\partial U}{\partial q_i} = 0 \quad , \quad i=1, \cdots, N$$

<u>Def:</u>

A function U is homogeneous of degree k if

$$U(t x_1, \cdots, t x_N) = t^k U(x_1, \cdots, x_N) \quad \forall t>0$$

For N-body problem

U is homogeneous of degree -1

⑤

Euler's theorem:

If U is homogeneous of degree $\overset{k}{-1}$ and differentiable, then

$$\sum_{i=1}^{N} x_i \frac{\partial U}{\partial x_i} = -U$$

$$(= kU)$$

According to Euler's theorem, we have

$$\sum_{i=1}^{N} q_i \frac{\partial U}{\partial q_i} = -U \qquad (+)$$

recall that U is always positive (according to its definition). Hence the eqn $(+)$ can not be satisfied for equilibrium solutions.

Namely, there are no equilibrium sols of NBP.

- Central Configuration

consider a typical type of sol.

$$q_i(t) = \phi(t)\, a_i \qquad , i = 1, \cdots, N$$

where a_i are constant vectors, $\phi(t)$ is scalar-valued function.

Substitute these into (*)

$$m_i \ddot{\phi} a_i = \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{G m_i m_j (\phi a_j - \phi a_i)}{\| \phi a_i - \phi a_j \|^3} \qquad i = 1, \cdots, N$$

$$\Downarrow$$

$$\underbrace{|\phi|^3 \phi^{-1} \ddot{\phi}}_{-\lambda \,:\, \text{const}} \underbrace{m_i a_i}_{\text{const}} = \underbrace{\sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{G m_i m_j (a_j - a_i)}{\| a_i - a_j \|^3}}_{\text{Const}} \qquad i = 1, \cdots, N$$

Since m_i, a_i are all constants, we then have

$$(C1) \qquad \ddot{\phi} = - \frac{\lambda \phi}{|\phi|^3}$$

where λ is also a constant, and

$$(C2) \qquad -\lambda m_i a_i = \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{G m_i m_j (a_j - a_i)}{\| a_i - a_j \|^3}, \quad i = 1, \cdots, N$$

Note, (C1) is one-dim Kepler problem, one solution is $\phi(t) = \alpha t^{2/3}$, $\alpha^3 = \frac{9\lambda}{2}$.

(C2) is difficult to solve, only for $N = 2, 3$ there are known complete sols. for $N > 3$, only special sols are known.

(6) Now consider $\underbrace{\text{planar}}$ N-body problem,

configuration space $\mathbb{R}^2$

Identify $\mathbb{R}^2$ with $\mathbb{C}$, $\mathbb{R}^2 \cong \mathbb{C}$

$$q_i \in \mathbb{C}, \quad p_i \in \mathbb{C}$$

We looking for sols like $q_i(t) = \phi(t)\, a_i$

with $a_i \in \mathbb{C}$ const and $\phi(t)$ ~~complexed~~ complex-valued function.

Multiplication by a complex number is equal to a rotation followed by a dilation or expansion, namely, a homography.

$\theta = \arg \phi(t)$

$$\phi(t) = \underbrace{|\phi(t)|}_{\text{modulo of } \phi(t)} \underbrace{e^{i\theta}}_{\text{rotation}}$$

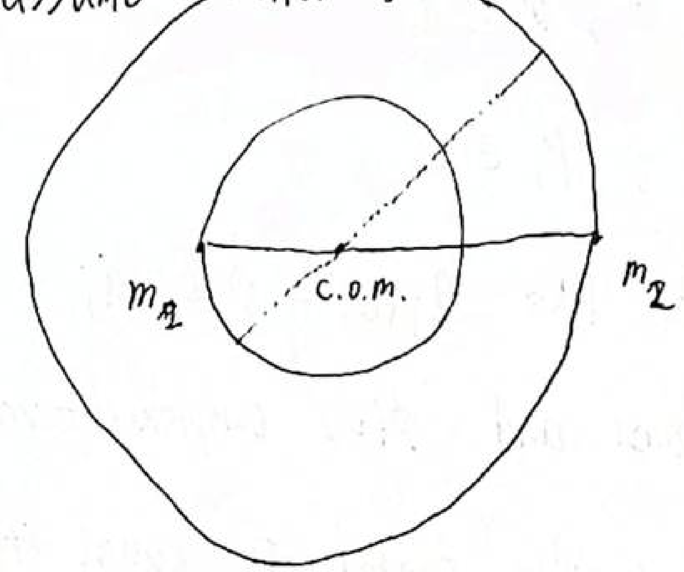
- > 1 expansion
- < 1 contraction
- $= 1$ only rotation

So in this case, we are looking for ~~sols~~ a sol. s.t. the configuration of particles is homografically equivalent to a fixed configuration $(a_1, \cdots, a_N)$

For instance (simplest case, $N=2$)

2-body problem, Circular motion, unit distance

assume center of mass is the origin



$$q_1 = -e^{it}\frac{m_2}{m_1+m_2}$$

$$q_2 = e^{it}\frac{m_1}{m_1+m_2}$$

We can also substitute the complex notation

$$q_i = \phi(t)\, a_i$$

into (*) and re-obtain eqns (C1), (C2)

However, in this case $\phi(t)$ is complex, so

eqn (C1) is a 2-dim Kepler problem.

Once you ~~can~~ ~~solve~~ have a sol $(a_1, \cdots, a_N)$ of eqn (C2),

you get a sol of N-body problem $q_i = \phi(t)\, a_i$

$\phi(t)$ can be circular, elliptic, parabolic, hyperbolic.

⑦ Def: (Central Configuration) c.c.

A configuration of the N particles given by constant vectors $a_1, \cdots, a_N$ satisfying (C2) for some λ is called a central configuration.

Rmk:

- Obviously, for a c.c., we have $q_i = \phi(t) a_i$

  $a_1, \cdots, a_N$ represents the shape, $\phi(t)$ represents the size.

- relative equilibrium : a_i are coplanar.

  (equilibrium in a rotating frame)

- any uniform scaling of a c.c. is also a c.c.

  (But the constanst will change correspondingly.)

moment of inertia

$$I = \frac{1}{2} \sum_{i=1}^{N} m_i \|q_i\|^2$$

It measures the size of the system.

Now, (C2) is equivalent to

$$\frac{\partial U}{\partial q}(a) + \lambda \frac{\partial I}{\partial q}(a) = 0$$

dot product of vector a (a is a c.c.)

$$\frac{\partial U}{\partial q}(a) \cdot a + \lambda \frac{\partial I}{\partial q}(a) \cdot a = 0$$

recall that U is homogeneous of degree -1

clearly I is homogeneous of degree 2

Apply Euler's theorem

$$-U(a) + \lambda 2I(a) = 0$$

$$\Rightarrow \quad \lambda = \frac{U(a)}{2I(a)} > 0$$

Note: · if a is c.c., then $\sum_{i=1}^{N} m_i a_i = 0$, namely, center of mass of c.c. is at origin.

· if A is an orthogonal matrix $(AA^T = I,\ A^T A = I)$ (3×3 or 2×2)

then $Aa = (Aa_1, \cdots, Aa_N)$ is c.c. with λ.

· If $\tau \neq 0$, then $(\tau a_1, \cdots, \tau a_N)$ is c.c. with λ/τ^3.

· configuration similar to c.c. is c.c.

· When counting c.c., one only counts similarity classes.

⑧ Totall Collapse

Lemma: Lagrange - Jacobi formula

$$\ddot{I} = 2T - U = T + h$$

where h is the energy of the system of N-bodies.

($H = h$)

Proof:

$$I = \frac{1}{2} \sum_{i=1}^{N} m_i \|q_i\|^2$$

$$\dot{I} = \sum_{i=1}^{N} m_i q_i \cdot \dot{q}_i \quad , \text{並}$$

$$\ddot{I} = \sum_{i=1}^{N} m_i \dot{q}_i \cdot \dot{q}_i + \sum_{i=1}^{N} m_i q_i \cdot \ddot{q}_i$$

$$\frac{\partial U}{\partial q_i}$$

$$= 2T + \sum_{i=1}^{N} q_i \cdot \frac{\partial U}{\partial q_i}$$

$$= 2T - U$$

$$= T + h.$$

∎

Lemma: (Sundman's inequality)

Let $c = \|A\|$ be the magnitude of angular momentum and $h = T - U$ the total energy of the system, then

$$c^2 \leq 4I(\ddot{I} - h)$$

Proof:

$$c = \|A\| = \|\sum m_i q_i \times \dot{q}_i\|$$

$$\leq \sum m_i \|q\| \|\dot{q}_i\|$$

$$= \sum \sqrt{m_i} \|q\| \sqrt{m_i} \|\dot{q}_i\|$$

Then apply Cauchy's inequality

$$c^2 \leq \sum m_i \|q\|^2 \sum m_i \dot{q}_i^2$$

$$= 2I \, 2T$$

$$= 4I(\ddot{I} - h)$$

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Theorem (Sundman's theorem on total collapse)

If total collapse occurs then angular momentum is zero and it will only take a finite amount of time.

That is, if $I(t) \to 0$ as $t \to t_1$ then

$$A = 0 \quad , \quad t_1 < \infty$$

⑨ Proof:

Let h be energy of the system, then $\ddot{I} = T + h$.

Assume $I(t)$ is defined $\forall t \geq 0$, and $I \to 0$ as $t \to \infty$.

Now $I \to 0$ means $q_i \to 0$, then $U \to \infty$,

due to conservation of the energy, $T \to \infty$.

Correspondingly, $\ddot{I} \to \infty$ as $t \to \infty$.

According to the definition of limit, $\exists\, t^* > 0$ s.t.

$\ddot{I} \geq 1 \quad \forall t \geq t^*$.

$\Downarrow$ Integration

$I(t) \geq \frac{1}{2} t^2 + at + b$, where a, b are constants.

However, if total collapse happens, we must have

$I \to 0$ as $t \to \infty$. Contradiction.

Now, assume $I \to 0$ as $t \to t_1^- < \infty$.

We still have $U \to \infty$, $T \to \infty$, and $\ddot{I} \to \infty$.

Again, by ~~via~~ the def. of limit, $\exists\, t_2 > 0$ s.t. $\ddot{I}(t) > 0$, for $t_2 \leq t < t_1$.

We then have, for $t_2 \leq t < t_1$

$I(t) > 0$, $\ddot{I}(t) > 0$, $I(t) \to 0$ as $t \to t_1$

We conclude that $\dot{I}(t) \leq 0$ (non-increasing) on $t_2 \leq t < t_1$.

Use Sundman's inequality, multiply by $-\dot{I}I^{-1} > 0$

$$c^2 \leq 4I\ddot{I} - 4Ih$$

$$-\frac{1}{4}c^2\dot{I}I^{-1} \leq h\dot{I} - \dot{I}\ddot{I}$$

Integrate this

$$-\frac{1}{4}c^2 \log I \leq hI - \frac{1}{2}\dot{I}^2 + \underset{\text{some constant}}{k}$$

$$\leq hI + k$$

therefore,

$$\frac{1}{4}c^2 \leq \frac{hI+k}{\log I^{-1}}$$

As $t \to t_1$, $I \to 0$, and $\frac{hI+k}{\log I^{-1}} \to 0$

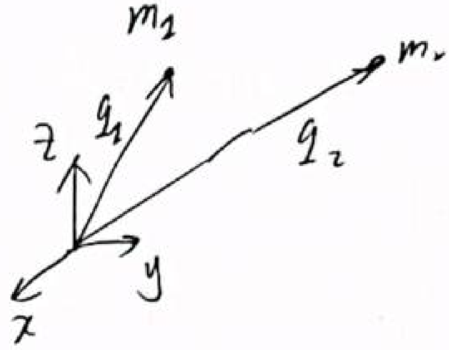
We get $c=0$, i.e., $A=0$ ☒

① Two - Body Problem

equations of motion:

$$m_1 \ddot{q}_1 = \mathcal{G} m_1 m_2 \frac{q_2 - q_1}{\| q_2 - q_1 \|^3}$$

$$m_2 \ddot{q}_2 = \mathcal{G} m_1 m_2 \frac{q_1 - q_2}{\| q_1 - q_2 \|^3}$$



$$\Rightarrow \quad m_1 \ddot{q}_1 + m_2 \ddot{q}_2 = 0 \qquad \text{center of mass}$$

$$\ddot{q}_1 - \ddot{q}_2 = - \mathcal{G}(m_1 + m_2) \frac{q_1 - q_2}{\| q_1 - q_2 \|^3}$$

$\Rightarrow \quad \ddot{C} = 0$, denote $u := q_1 - q_2$

$\mu := \mathcal{G}(m_1 + m_2)$

we obtain $\quad \ddot{u} = - \dfrac{\mu u}{\| u \|^3}$ Kepler problem

1st integrals: $C = L_0 t + C_0$.

To understand more about the problem, we introduce the Jacobi coordinates (g, u, G, v)

$$g = \nu_1 q_1 + \nu_2 q_2 \qquad G = p_1 + p_2$$

$$u = q_2 - q_1 \qquad v = -\nu_2 p_1 + \nu_1 p_2$$

where

$$\nu_1 = \frac{m_1}{m_1+m_2}, \quad \nu_2 = \frac{m_2}{m_1+m_2}$$

$$\nu = m_1+m_2, \quad M = \frac{m_1 m_2}{m_1+m_2}$$

and

g : center of mass, G : total linear momentum

u : relative position of m_2 with respect to m_1

v : a scaled momentum.

The map from (q_1, q_2, p_1, p_2) to (g, u, G, v) is linearly symplectic, which preserves the Hamiltonian character.

The Hamiltonian in (q, p) is

$$H = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} - \frac{\mathcal{G} m_1 m_2}{\|q_1 - q_2\|}$$

and in (g, u, G, v) is

$$H = \frac{\|G\|^2}{2\nu} + \frac{\|v\|^2}{2M} - \frac{\mathcal{G} m_1 m_2}{\|u\|}$$

Note: $\mathcal{G}$ is the gravitational constant, and G is a variable. They look similar.

② Two-Body Problem

the eqns of motion in (g, u, G, v) is

$$\dot{g} = \frac{\partial H}{\partial G} = \frac{G}{\nu} \quad , \quad \dot{G} = -\frac{\partial H}{\partial g} = 0$$

$$\dot{u} = \frac{\partial H}{\partial v} = \frac{v}{M} , \qquad \dot{v} = -\frac{\partial H}{\partial u} = -\frac{\mathcal{G} m_1 m_2 u}{\|u\|^3}$$

Obviously, G = constant is a 1st integral.

total linear momentum

g moves with constant linear velocity

center of mass

If we take $g = 0$, $G = 0$, then we have ~~u, v~~

the problem related to u, v.

Furthermore, the eqns reduce to

$$\ddot{u} = -\frac{\mathcal{G}(m_1 + m_2) u}{\|u\|^3}$$

This is just the central force problem or Kepler problem.

The motion of one body ~~m_2~~ B, when viewed from another body ~~m_1~~ A, is as if the body A were a fixed body with mass $m_1 + m_2$ and the body B were attracted to the body A by a central force.

The Kepler problem

Consider a two-body problem : the first body is massive s.t. its position is fixed and the second body is mass 1.

namely, $m_1 >> m_2 = 1$

q_1 fixed

Therefore, eqns of motion of the second body ~~is~~ are

$$(K1) \qquad \ddot{q} = -\frac{\mu q}{\|q\|^3} \quad , \quad q \in \mathbb{R}^3$$

q : position vector of the second body

$\mu = g m_1$, m_1 is the mass of the first body fixed at the origin.

defin~~ing~~ ing $p = \dot{q}$, we have the Hamiltonian

$$(K2) \qquad H = \frac{\|p\|^2}{2} - \frac{\mu}{\|q\|}$$

and Hamilton's eqn:

$$\dot{q} = p \quad , \quad \dot{p} = -\frac{\mu q}{\|q\|^3}$$

(K1) and (K2) define the Kepler problem.

(3) Note that 2-body problem ~~is~~ can be reduced to this problem.

Kepler problem = central force problem with $V(r) \propto \frac{1}{r}$

For kepler problem, we don't care about center of mass.

Indeed, the center of mass is very close to ~~m_1~~ position of m_1.

Angular momentum $A = q \times p$, and $A_1 = q_2 p_3 - q_3 p_2$

$$\dot{A}_1 = \{A_1, H\} = \frac{\partial A_1}{\partial q} \cdot \frac{\partial H}{\partial p} - \frac{\partial A_1}{\partial p} \cdot \frac{\partial H}{\partial q}$$

$$= p_3 p_2 + (-p_2) p_3 - \left(-q_3 \cdot \frac{\mu q_2}{\|q\|^3} + q_2 \cdot \frac{\mu q_3}{\|q\|^3} \right)$$

$$= 0$$

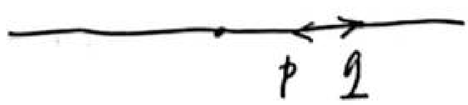
Indeed, $\dot{A} = 0$ (we have three 1st integrals)

If $A = 0$, we have $q \times p = 0$, namely

q and p are parallel

motion is collinear

or



then

$$\frac{d}{dt}\left(\frac{q}{\|q\|}\right) = \frac{\dot{q}\,\|q\| - q\,\frac{q\cdot\dot{q}}{\|q\|}}{\|q\|^2}$$

$$= \frac{1}{\|q\|^3}\left(\dot{q}\,(q\cdot q) - q\,(q\cdot\dot{q})\right)$$

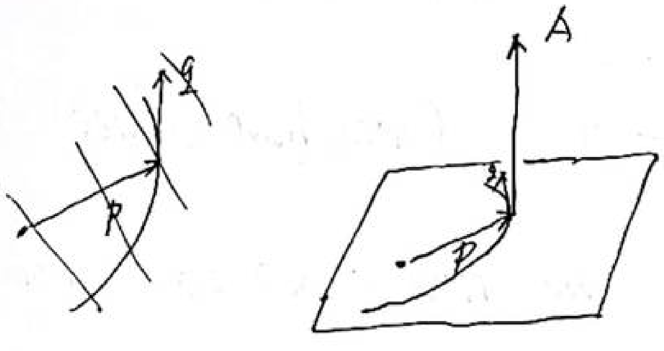
$$= \frac{-1}{\|q\|^3}\left(q \times \underbrace{(q \times \dot{q})}_{A}\right) = 0$$

$$u \times (v \times w) = (u\cdot w)\,v - (u\cdot v)\,w$$

this means $\frac{q}{\|q\|}$ is a constant vector, obvious since it is just a unit vector.

If $A \neq 0$, then $q \perp A$, $p \perp A$

$A = q \times p$



Since A is a constant of motion, the motion takes place in the plane orthogonal to A

↑

invariant plane

Without loss of generality, we can take A to be parallel with z-axis.

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The problem is then reduced to

$$(\text{K3}) \qquad \ddot{q} = -\frac{\mu q}{\|q\|^3}, \quad q \in \mathbb{R}^2$$

which is 2 dof with one integral, hence is solvable "up to quadrature".

Now consider polar coordinates

$$q = (r\cos\theta,\ r\sin\theta,\ 0)$$

$$\Rightarrow \quad \dot{q} = (\dot{r}\cos\theta - r\sin\theta\,\dot{\theta},\ \dot{r}\sin\theta + r\cos\theta\,\dot{\theta},\ 0)$$

which gives rise to

$$(\text{K4}) \quad A = q \times \dot{q} = (0,\ 0,\ r\dot{r}\sin\theta\cos\theta + r^2\cos^2\theta\,\dot{\theta}$$

$$- r\dot{r}\sin\theta\cos\theta + r^2\sin^2\theta\,\dot{\theta})$$

$$= (0,\ 0,\ r^2\dot{\theta})$$

$$= (0,\ 0,\ c) \quad \text{where } c \in \mathbb{R} \text{ is some constant.}$$

Indeed, $\frac{1}{2} r r\dot{\theta}$ is equal to the rate at which area is swept out by a radius vector

the triangle

the area $\frac{1}{2} r \cdot r d\theta$

$\Rightarrow$ the particle sweeps out area at a constant rate of $\frac{c}{2}$.

「at unit time the particle sweeps out equal area」

「Kepler's second law」

Now we solve the kepler problem

$$\mu \frac{d}{dt}\left(\frac{q}{\|q\|}\right) = A \times \frac{-\mu q}{\|q\|^3} = A \times \dot{p}$$

Integrating this

(K5) $\mu\left(\vec{e} + \frac{q}{\|q\|}\right) = p \times A$

where $\vec{e}$ is a vector integration constant.

Clearly $\vec{e} \perp A$ (i.e. $\vec{e} \cdot A = 0$), $\vec{e}$ lies in the invariant plane defined before.

If $A = 0$, then $\vec{e} = -\frac{q}{\|q\|}$, $\vec{e}$ lies ~~in~~ on the line of motion and $\vec{e}$ is a unit vector.

(k6) $$\vec{e} = \frac{p \times A}{\mu} - \frac{q}{\|q\|}$$

We call $\vec{e}$ the eccentricity vector.

$\mu\vec{e}$ is called $\underbrace{\text{Laplace-Runge-Lenz}}_{\text{(Hermann-Bernoulli-)}}$ vector.

⑤ ~~From~~ We consider $A \neq 0$

we have easily have

$$\mu\left(\vec{e}\cdot q + \|q\|\right) = q \cdot p \times A$$

$$\Rightarrow \mu\left(\vec{e}\cdot q + \|q\|\right) = A \cdot A$$

$$\Rightarrow \vec{e}\cdot q + \|q\| = \frac{c^2}{\mu} \qquad (K7)$$

~~and~~
If $\vec{e} = 0$, we obtain $\|q\| = \frac{c^2}{\mu}$. Circular motion

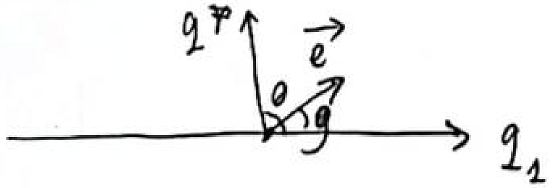
recall that $r^2\dot{\theta} = c$, and $\|q\| = r$, from which

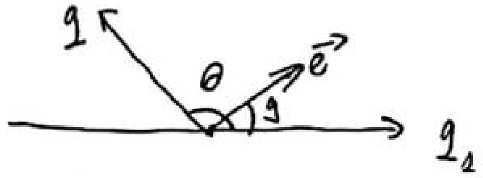
we get $\frac{c^4}{\mu^2}\dot{\theta} = c \Rightarrow \dot{\theta} = \frac{\mu^2}{c^3}$

Therefore, for $\vec{e} = 0$, ~~we~~ the particle moves on a circle with uniform angular velocity.

If $\vec{e} \neq 0$, we denote $e := \|\vec{e}\|$ (which is also a constant)

Since $\vec{e}$ is a constant, ~~ass~~ let's assume that the angle from the positive q_1 axis to $\vec{e}$ is "g"





$q = (r\cos\theta, r\sin\theta, 0)$

denote $f := \theta - g$, then

$$\vec{e} \cdot g = \|e\| \, \|g\| \cos(\theta - g)$$

$$= e r \cos(f)$$

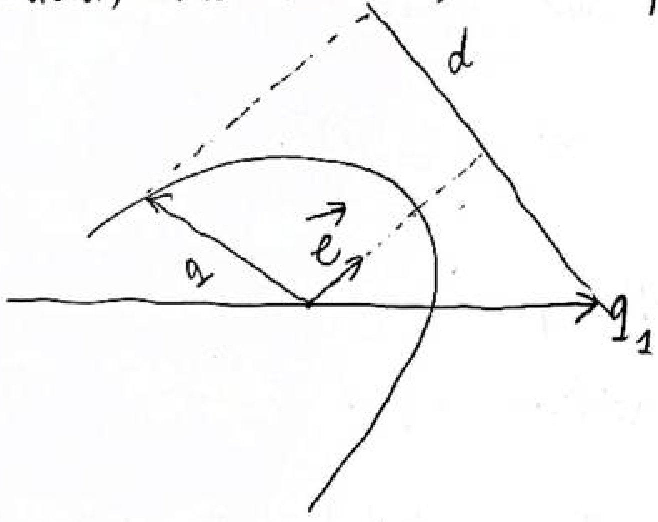
then from previous eqn (K7)

$$e r \cos f + r = \frac{c^2}{\mu}$$

$$\Rightarrow \quad r = \frac{c^2/\mu}{1 + e\cos f} = \frac{c^2/\mu}{1 + e\cos(\theta - g)} \quad (K8)$$

(expression of the orbit in polar coordinates)

Indeed, this is exactly the expression of a conic section



the distance from the origin to the line d is $\cancel{\frac{c^2}{\mu e}}$ $c^2/\mu e$

We rewrite the eqn to obtain

$$r = e \cancel{\delta} \left(\frac{c^2}{\mu e} - r \cos f \right) \quad (K9)$$

the distance of the particle from the origin is equal to e times its distance from the line d.

Kepler's 2nd law

$\Rightarrow$ conic section of eccentricity e with one focus at the origin.

⑥ Recall that

- $0 < e < 1$: ellipse
- $e = 1$: parabola — this is the case if $A \neq 0$
- $e > 1$: hyperbola

angle between q and $\vec{e}$

Note that ~~we~~ when $f = 0$, the particle is at the point closest to the origin. This indicates that $\vec{e}$ ~~p~~ points to the point of closest approach. 「$f = \angle(\vec{e}, \vec{q})$」

This point is called pericenter. 近中心点

(If the origin is the Sun, it is called perihelion; 近日点
Earth perigee) 近地点

(By the way, apo- 远)

The angle g is called the argument of the pericenter.

近中心点辐角

f is called true anomaly. 真近点角

Let $h = \frac{\dot{q}^2}{2} - \frac{\mu}{\|q\|}$ be the energy of the particle (a conserved quantity)

then we have

$$\mu^2(e^2 - 1) = 2hc^2 \qquad (K10)$$

Indeed, $\left[\mu\left(\vec{e} + \frac{q}{\|q\|}\right)\right]^2 = [p \times A]^2$

recall $A = q \times p$
so $p \perp A$

take the square, we obtain

$$\mu^2\left(\vec{e}^{\,2} + \overset{1}{\cancel{\|q\|}} + 2\left\langle \vec{e}, \frac{q}{\|q\|} \right\rangle\right) = p^2 A^2$$

$$\mu^2\left(e^2 + 1 + 2\left\langle \vec{e}, \frac{q}{\|q\|} \right\rangle\right) = \dot{q}^2 c^2$$

we use (K7) to get

$$\mu^2\left(1 + e^2 + \frac{2}{r}\left(\frac{c^2}{\mu} - r\right)\right) = \dot{q}^2 c^2$$

$$\mu^2\left(1 + e^2 + \frac{2c^2}{r\mu} - 2\right) = \dot{q}^2 c^2$$

$$\mu^2(e^2 - 1) = \dot{q}^2 c^2 - \frac{2\mu}{r} c^2 = 2hc^2.$$

From this, we have the following table for $c \neq 0$

e	h	orbit
<1	<0	ellipse
$=1$	$=0$	parabola
>1	>0	hyperbola

(7) Now we focus on the elliptic orbit $e<1$, $h<0$

We know that the area of ellipse is

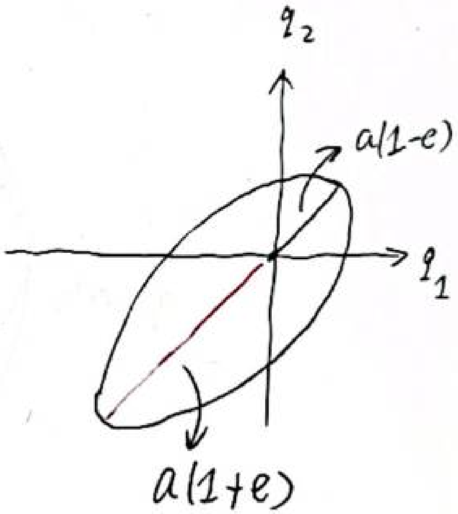
$$\pi ab$$

Where a is the semi-~~maj~~ major axis, b is the ~~minor~~ ~~major~~ semi-minor axis. $b = a\sqrt{1-e^2}$

We also know that the ~~unit~~ out area swept by the radius vector is equal to $\frac{1}{2}r^2\dot{\theta} = \frac{1}{2}c$.

If the period of elliptic orbit is T, then area of the ellipse is

$$\frac{1}{2}Tc$$



Therefore,

$$\pi a^2\sqrt{1-e^2} = \frac{1}{2}Tc.$$

Besides, we have $r = \dfrac{c^2/\mu}{1+e\cos f}$

if $f=0$, we have $r(0) = \dfrac{c^2/\mu}{1+e} = a(1-e)$

$$\Rightarrow c^2/\mu = a(1-e^2) \quad (K*)$$

Eventually, we obtain:

$$(K11) \qquad \frac{4\pi^2}{\mu} = \frac{T^2}{a^3}. \qquad \text{Kepler's 3rd law}$$

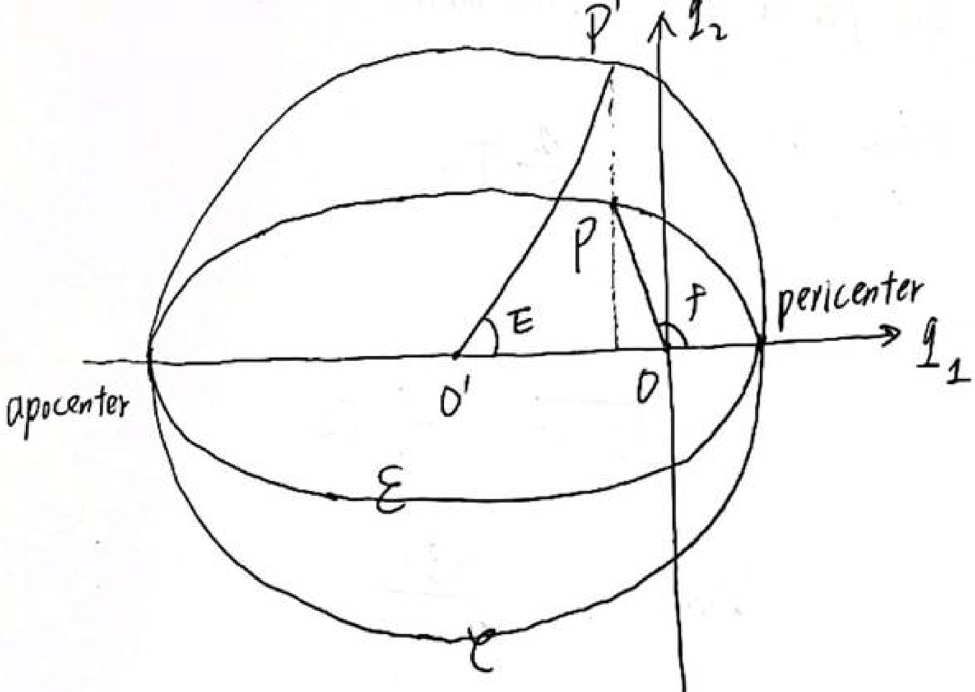
Anomalies and Kepler's equation

We assume, to ~~simplify our life~~ make our lives easier, that the orbit is an ellipse, i.e. $h<0$ ~~and~~ and $c \neq 0$.
(not circular)

By suitably choose axes, we can assume $\overset{g}{\varpi}=0$, then

$$r(f) = \frac{a(1-e^2)}{1+e\cos f}(\cos f, \sin f) \quad f \in \mathbb{R}$$

is the coordinates of the body in xy-plane



O is the focus of the ellipse, f is true anomaly

$\mathcal{C}$ is a circle, tangent to $\mathcal{E}$ at pericenter and apocenter, centered at $(-ae, 0)$ with radius a.

④ E: angle ~~between~~ from x-axis to $P'O$, note that $P'P \perp x$-axis.

↳ eccentric anomaly 偏近点角

If coords of P is (q_1, q_2), then we have that

$$(K12) \quad q_1 = a(\cos E - e), \quad q_2 = \underbrace{a\sqrt{1-e^2}}_{b} \sin E$$

Indeed, after simple calculation, we find

$$\left(\frac{q_1 + ae}{a}\right)^2 + \left(\frac{q_2}{b}\right)^2 = 1$$

Which means (q_1, q_2) is located at the ~~an~~ ellipse $\mathcal{E}$ whose center is $(-ae, 0)$

From $r(f) = (q_1, q_2)$ we obtain the relationship between f and E

$$(K13) \quad \cos f = \frac{\cos E - e}{1 - e\cos E}, \quad \sin f = \frac{\sqrt{1-e^2}\sin E}{1 - e\cos E}$$

alternatively, the following is satisfied

$$(K14) \quad r = a(1 - e\cos E)$$

$$(K15) \quad \tan\frac{f}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2}$$

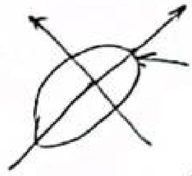
let T be the orbital period, define

$$(K16) \qquad n := \frac{2\pi}{T} \qquad \overset{\text{平运动}}{\text{mean motion}} \quad (\text{frequency})$$

and let t_0 be the time of passage at pericenter, i.e. $r(t_0) = r_{min}$

define $\quad M := n(t - t_0)$ mean anomaly
平近点角

(K17)



at t_0, the body is at the pericenter

Kepler equation $\qquad M = E - e\sin E \qquad (K18)$

Indeed, $\quad r(t) = \left(a(\cos E - e),\ a\sqrt{1-e^2}\sin E\right)$

$$\dot r(t) = \left(-a\dot E\sin E,\ a\sqrt{1-e^2}\cos E\,\dot E\right)$$

~~所~~ we know that $\quad r \times \dot r = (0, 0, c)$ ← angular momentum

therefore

$$c = a^2\sqrt{1-e^2}\,\dot E\left(\cos E(\cos E - e) + \sin^2 E\right)$$

$$= a^2\sqrt{1-e^2}\,\dot E\,(1 - e\cos E)$$

By using $c^2/\mu = a(1-e^2)$ we get

$$\frac{\sqrt{\mu}}{a^{3/2}} = (1 - e\cos E)\dot E$$

then Kepler's 3rd law gives $\qquad \underset{n=}{\frac{2\pi}{T}} = (1 - e\cos E)\dot E$

⑨ Integrate from t_0 to t, we obtain (not $E(t=0)=0$ i.e. $E_0 = 0$.)

$$n(t-t_0) = E - e\sin E$$

Remark:

a: semi-major axis

e: eccentricity

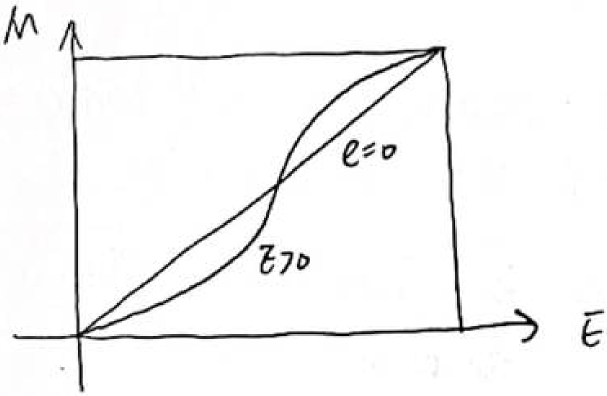
ω: argument of pericenter

M: mean anomaly

orbital elements (for planar problem)
轨道根数

$$\frac{dM}{dE} = 1 - e\cos E > 0,$$ hence Kepler equation

determines a unique and smooth function E wrt. M.



Three-body Problem

the Hamiltonian of three-body problem is

$$H = \sum_{i=1}^{3} \frac{\|p_i\|^2}{2m_i} - \sum_{1 \le i < j \le 3} \frac{G\, m_i m_j}{\|q_i - q_j\|} \quad \text{, assume } G = 1$$

degree of freedom : 9

no. of equations of motion (first-order) : 18

1st integrals: 3 for linear momentum, 3 for c.o.m.

3 for angular momentum, 1 for energy

Jacobi coords for 3-body problem

(i) one coordinate locates ~~at~~ the center of mass;
(can be set to zero and q ignored)

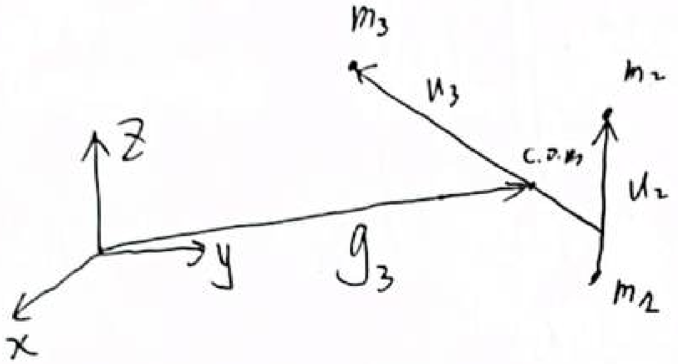
(ii) one coordinate is the vector from one ~~parti~~ body to another;
(useful when two bodies are close)

(iii) one coordinate is the vector from the c.o.m of first 2 bodyies

to the 3rd ~~m~~ body; (useful when one body is far away from the others.)

10

This procedure can be iterated to apply for N-body case



c.o.m of m_1, m_2 is

$$\frac{m_1 q_1 + m_2 q_2}{m_1 + m_2}$$

We choose center of mass of 3 bodies as

$$c = \frac{m_1 q_1 + m_2 q_2 + m_3 q_3}{m_1 + m_2 + m_3}$$

$$\bar{m} := m_1 + m_2 + m_3$$

$$u_2 = q_2 - q_1 \qquad M_2 := \frac{m_1 m_2}{m_1 + m_2}$$

$$u_3 = q_3 - \frac{m_1 q_1 + m_2 q_2}{m_1 + m_2} \qquad M_3 := \frac{m_3 (m_1 + m_2)}{m_1 + m_2 + m_3}$$

$$g_3 = \frac{m_1 q_1 + m_2 q_2 + m_3 q_3}{m_1 + m_2 + m_3} \qquad \alpha_0 := \frac{m_2}{m_1 + m_2}$$

$$\alpha_1 := \frac{m_1}{m_1 + m_2}$$

$$\begin{pmatrix} u_2 \\ u_3 \\ g_3 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 0 \\ -\alpha_1 & -\alpha_0 & 1 \\ \frac{\alpha_1 M_3}{m_3} & \frac{\alpha_0 M_3}{m_3} & \frac{M_3 \alpha_0 \alpha_1}{M_2} \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}$$

correspondingly, we have conjugate momentum

$$\begin{pmatrix} v_2 \\ v_3 \\ G_3 \end{pmatrix} = \begin{pmatrix} -\alpha_0 & \alpha_1 & 0 \\ -\frac{m_3}{\bar{m}} & -\frac{m_3}{\bar{m}} & \frac{m_1+m_2}{\bar{m}} \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}$$

after a long calculation, one finds

$$H = \frac{\|G_3\|^2}{2\bar{m}} + \frac{\|v_2\|^2}{2M_2} + \frac{\|v_3\|^2}{2M_3} - \frac{m_1 m_2}{\|u_2\|} - \frac{m_1 m_3}{\|u_3 + \alpha_0 u_2\|} - \frac{m_2 m_3}{\|u_3 - \alpha_1 u_2\|}$$

obviously, H is independent of g_3, c.o.m

$\Downarrow$

G_3 is an integral (total linear momentum)

To simplify the ~~prob~~ Hamiltonian, we can set

$$g_3 = 0, \quad G_3 = 0$$

then we obtain a Hamiltonian of 6 dof

(Jacobi reduction)

(11)

Now we introduce polar coords (r, θ, R, Θ) (for planar case)

$$u_2 = (r_1 \cos\theta_1,\ r_1 \sin\theta_1) \qquad u_3 = (r_2 \cos\theta_2,\ r_2 \sin\theta_2)$$

correspondingly, the conjugate momentum ~~is~~ are

$$v_2 = \left(R_1 \cos\theta_1 - \left(\frac{\Theta_1}{r_1}\right)\sin\theta_1,\ R_1 \sin\theta_1 + \left(\frac{\Theta_1}{r_1}\right)\cos\theta_1\right)$$

$$v_3 = \left(R_2 \cos\theta_2 - \left(\frac{\Theta_2}{r_2}\right)\sin\theta_2,\ R_2 \sin\theta_2 + \left(\frac{\Theta_2}{r_2}\right)\cos\theta_2\right)$$

r, θ : polar coords

R, Θ: radial and angular momenta

} symplectic coords

Eventually, the Hamiltonian will be

$$H = \frac{1}{2M_2}\left(R_1^2 + \left(\frac{\Theta_1}{r_1}\right)^2\right) + \frac{1}{2M_3}\left(R_2^2 + \left(\frac{\Theta_2}{r_2}\right)^2\right) - \frac{m_1 m_2}{r_1}$$

$$- \frac{m_1 m_3}{\sqrt{r_2^2 + \alpha_0^2 r_1^2 + 2\alpha_0 r_1 r_2 \cos(\theta_2 - \theta_1)}}$$

$$- \frac{m_2 m_3}{\sqrt{r_2^2 + \alpha_1^2 r_1^2 - 2\alpha_1 r_1 r_2 \cos(\theta_2 - \theta_1)}}$$

we introduce a set of symplectic coords again: (without changing r, R)

$$\varphi_1 = \theta_2 - \theta_1 \qquad \varphi_2 = -\theta_2 + 2\theta_1$$

$$\Phi_1 = 2\Theta_2 + \Theta_1 \qquad \Phi_2 = \Theta_2 + \Theta_1$$

then

$$H = \frac{1}{2M_2}\left(R_1^2 + (2\Phi_2 - \Phi_1)^2 / r_1^2\right)$$

$$+ \frac{1}{2M_3}\left(R_2^2 + (\Phi_1 - \Phi_2)^2 / r_2^2\right) - \frac{m_1 m_2}{r_1}$$

$$- \frac{m_1 m_3}{\sqrt{r_2^2 + \alpha_0^2 r_1^2 + 2\alpha_0 r_1 r_2 \cos(\phi_1)}}$$

$$- \frac{m_2 m_3}{\sqrt{r_2^2 + \alpha_1^2 r_1^2 - 2\alpha_1 r_1 r_2 \cos(\phi_1)}}$$

which is independent of $\phi_2 \Rightarrow \Phi_2 = \Theta_2 + \Theta_1$ is a 1st integral.

which is of 3 dof, with Φ_2 as parameter.

One can easily verifyp that $\Phi_2 = u_2 \times v_2 + u_3 \times v_2$ (angular momentum)

Heliocentric Coordinates. (for planetary $(N+1)$-Body problem)

Example: the Solar system

Sun Mercury Venus Earth Mars Jupiter Saturn Uranus Neptune

⑫ planetary $(N+1)$-body problem

one massive central body (a star) : m_0

N smaller bodies (planets) : $m_1, \cdots, m_N$

Usually, $m_0 \gg m_i \quad (i=1, \cdots, N)$

e.g. $\dfrac{\text{mass of Jupiter}}{\text{mass of Sun}} \approx 0.001$

heliocentric: the Sun/star will be the center of the system (origin of the coords)

Coordinates: $q_0, q_1, q_2, \cdots, q_N$ positions

$p_0, p_1, p_2, \cdots, p_N$ momenta

Heliocentric coords:

$$Q_0 := q_0 \qquad P_0 := p_0 + p_1 + \cdots + p_N$$

$$Q_i := q_i - q_0 \qquad P_i := p_i \qquad i = 1, \cdots, N$$

$$\begin{pmatrix} Q_0 \\ Q_1 \\ \vdots \\ Q_N \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & & 0 \\ -1 & 1 & & \\ \vdots & & \ddots & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}}_{A} \begin{pmatrix} q_0 \\ q_1 \\ \vdots \\ q_N \end{pmatrix}, \quad \begin{pmatrix} P_0 \\ P_1 \\ \vdots \\ P_N \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 & \cdots & 1 \\ 0 & 1 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & & 1 \end{pmatrix}}_{(A^T)^{-1}} \begin{pmatrix} p_0 \\ p_1 \\ \vdots \\ p_N \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} 1 & 0 & 0 & & 0 \\ 1 & 1 & & 0 & \vdots \\ & & \ddots & \ddots & \vdots \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The original Hamiltonian is

$$H = \sum_{i=0}^{N} \frac{\| p_i \|^2}{2m_i} - \sum_{0 \le i < j \le n} \frac{m_i m_j}{\| q_i - q_j \|}$$

With new coords, we have

$$q_i - q_j = Q_i - Q_j \qquad i, j \neq 0$$

$$p_0 = P_0 - P_1 - \cdots - P_N$$

thum

$$\frac{\| p_0 \|^2}{2m_0} = \frac{\| P_0 - P_1 - \cdots - P_N \|^2}{2m_0} \qquad - \frac{1}{2m_0} 2 \langle P_0, \sum_{i=1}^{N} P_i \rangle$$

$$= \frac{\| P_0 \|^2}{2m_0} + \frac{1}{2m_0} \sum_{i=1}^{N} \| P_i \|^2 + \frac{1}{2m_0} \sum_{1 \le i < j \le N} 2 \langle P_i, P_j \rangle$$

therefore, the planetary Hamiltoniam is

$$H_{plt} = \frac{\| P_0 \|^2}{2m_0} + \sum_{i=1}^{N} \left(\frac{1}{2m_0} + \frac{1}{2m_i} \right) \| P_i \|^2 - \sum_{i=1}^{N} \frac{m_0 m_i}{\| Q_i \|}$$

$$- \frac{1}{m_0} \langle P_0, \sum_{i=1}^{N} P_i \rangle + \frac{1}{m_0} \sum_{1 \le i < j \le N} \langle P_i, P_j \rangle$$

$$- \sum_{1 \le i < j \le N} \frac{m_i m_j}{\| Q_i - Q_j \|}$$

Now, since P_0 is the total linear momentum.

by assuming $P_0 = 0$, one gets

$$H_{plt} = \sum_{i=1}^{N}\left(\frac{\|P_i\|^2}{2M_i} - \frac{M_i \overline{m_i}}{\|Q_i\|}\right) + \sum_{1\le i<j\le N}\left(\frac{\langle P_i, P_j\rangle}{m_0} - \frac{m_i m_j}{\|Q_i - Q_j\|}\right)$$

$$= H^{(0)}_{plt} + H^{(1)}_{plt}$$

where $M_i = \dfrac{m_0 m_i}{m_0 + m_i}$, $\overline{m_i} = m_0 + m_i$, $i = 1, \cdots, N$

the integrable part $H^{(0)}_{plt}$ is the N uncoupled kepler problems with M_i as mass of fixed center

$\overline{m_i}$ as mass of the planet

the perturbation part $H^{(1)}_{plt} = H^{(1,c)}_{plt} + H^{(1,P)}_{plt}$

complementary part due to change of coords

principal part related to mutual distance

if $m_0 \gg m_i$, then $H^{(1)}_{plt}$ is small, and the ~~op~~ unperturbed ellipses are well separated.

The angular momentum

$$A = \sum_{i=0}^{N} Q_i \times P_i = \sum_{i=0}^{N} Q_i \times P_i = \sum_{i=1}^{N} Q_i \times P_i$$

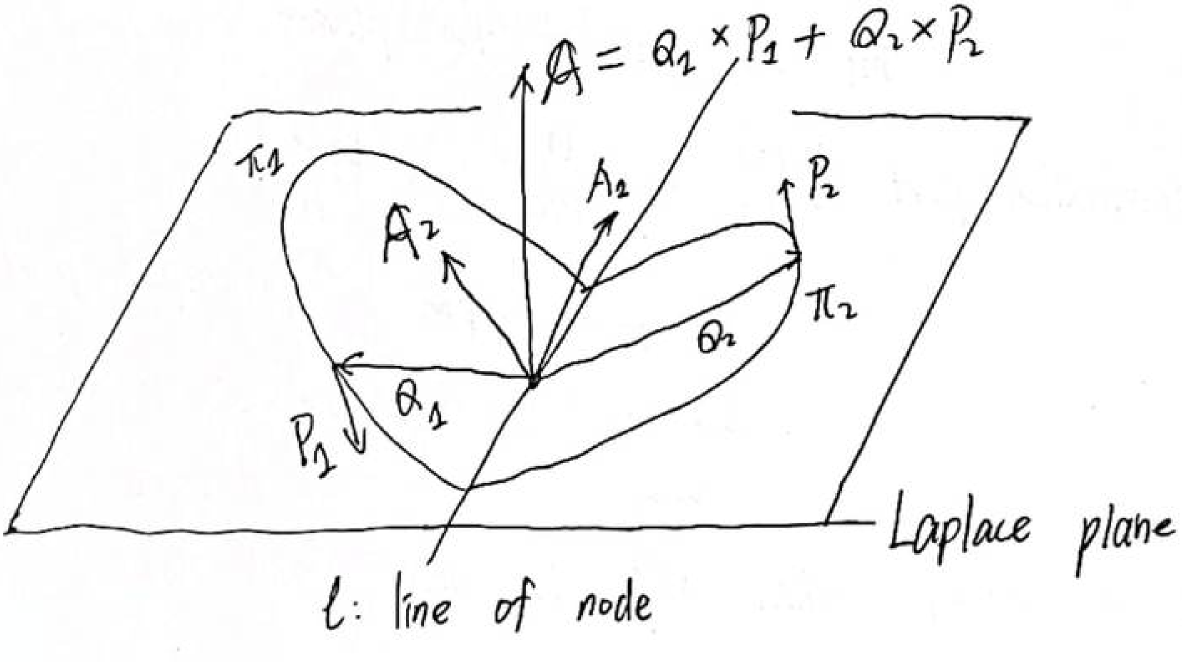
If $A \neq 0$, the plane orthogonal to A is call invariant Laplace plane.

Note we now have N ~~uncoupled Kepler problem plus~~ ficitious bodies perturbation, with positions Q_i and momentum P_i

denote $A_i := Q_i \times P_i$

Consider $N=2$, then $A_1 = Q_1 \times P_1$, $A_2 = Q_2 \times P_2$

the Laplace plane and the two planes spanned by (Q_1, P_1) and (Q_2, P_2) have a common intersection, called line of node.



$l \perp A_1$, $l \perp A_2$ since l is the intersection of π_1 and π_2, and $A_1 \perp \pi_1$, $A_2 \perp \pi_2$

$A = A_1 + A_2 \Rightarrow A \perp l \Rightarrow l$ is in Laplace plane.

(14) Delaunay variables:

We consider the following two-body problem

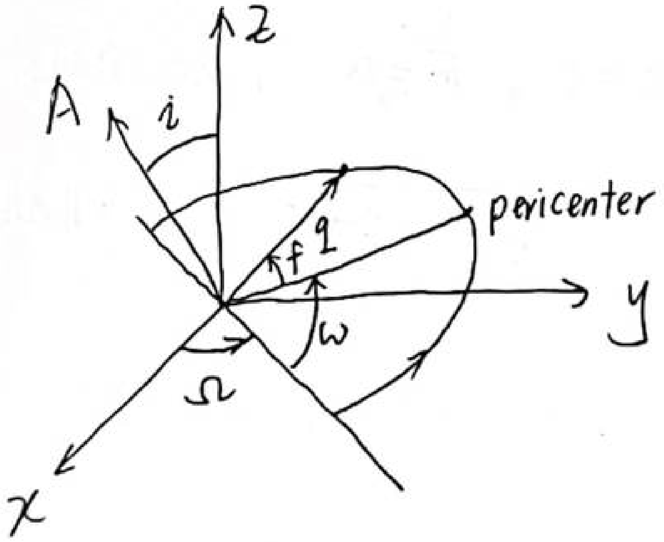
$$H = \frac{\|P\|^2}{2M} - \frac{M\bar{m}}{\|q\|}$$

where $P \in \mathbb{R}^3$, $q \in \mathbb{R}^3 \setminus \{0\}$.

This is integrable, and we already know the solution of it.

We now introduce Delaunay variables for two-body problem with negative energy.

$H = h < 0 \Rightarrow$ motion is elliptical, $0 < e < 1$.



Delaunay node: $\bar{\nu} := \hat{z} \times A$ which is the intersection of xy plane and orbital plane

$(\hat{x}, \hat{y}, \hat{z})$ is the basis of $\mathbb{R}^3$.

The Delaunay action-angle variables

$\Lambda := M\sqrt{\bar{m}\, a}$ $\qquad$ $\ell := M$ mean anomoly

$\Gamma := \Lambda\sqrt{1-e^2}$ $\qquad$ $g := \omega$ argument of perihelion

$\Theta := \Gamma \cos i$ $\qquad$ $\theta := \Omega$ longitude of accending node

Obviously, Delaunay variables are related to orbital elements.

- Γ is equal to length of angular momentum

$\Gamma = \|A\|$

Indeed, if $M = 1$, $\bar{m} = \mu$, we have

$$\Gamma = \sqrt{\mu a}\sqrt{1-e^2} = c = \|A\|$$

- $\Theta = \|A_3\|$ if $A = (A_1, A_2, A_3)$

- Delaunay vars are singular when A is vertical (node is not well-defined) and when the orbit is circular (perihelion is not unique)

(15) With $(\Lambda, \Gamma, \Theta, l, g, \theta)$ the Hamiltonian is

$$H(\Lambda) = -\frac{M^3 \bar{m}^2}{2\Lambda^2}$$

① Restricted Three-Body Problem

$N=3$

three bodies, m_0, m_1, $m=0$

~~m_1~~ m_0 and m_1 constitutes a 2-body system

m moves under the gravitational influence of m_0 and m_1.

position vectors : $q_0, q_1, q \in \mathbb{R}^3$

the eqns of motion related to $q_2 - q_1$ or $q_1 - q_2$ is

let $u := q_2 - q_1$

$$\ddot{u} = -\frac{G(m_1 + m_0)u}{\|u\|^3}$$

Which means the relative motion of P_1 wrt P_0 is known.

Motion of m:

$$m\dot{q} = -\frac{Gmm_0(q-q_0)}{\|q-q_0\|^3} - \frac{Gmm_1(q-q_1)}{\|q-q_1\|^3}.$$

To better understand the problem, we will assume the motion of

P_1 wrt P_0 is circular, namely $\|u\| = 1$, the radius vector

is of unit length. $\|q_0 - q_1\| = 1$

We also assume that $G = 1$

$$\mu \triangleq G(m_0 + m_1) = 1,$$

in this case, the mean motion of the two-body system is equal to 1, i.e., $n_1 = 1$

Let the coordinates of the asteroid m in the inertial, or sidereal, system, be $q = (X, Y, Z)$.

let $m_0 = 1-\mu$, then $m_1 = \cancel{1+}\mu$, consider the origin is located at the barycenter m_0 and m_1. The motion of m_0 and m_1 can be

$$q_0 = (-\mu \cos t, -\mu \sin t)$$

$$q_1 = ((1-\mu)\cos t, (1-\mu)\sin t)$$

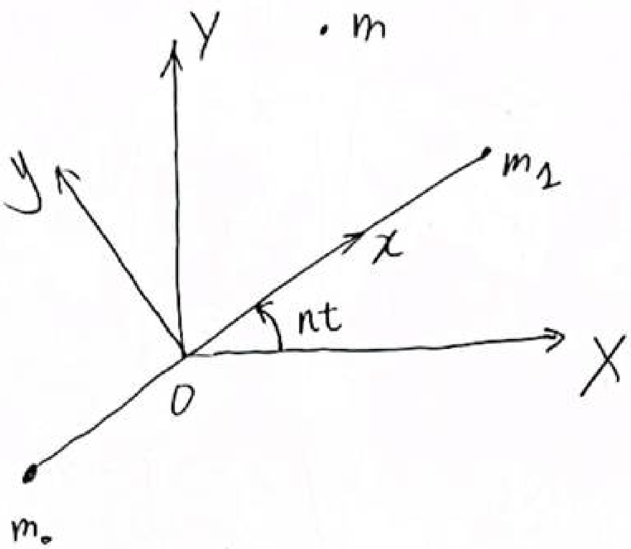
then the eqns of motion are

$$\ddot{q} = -\frac{(1-\mu)(q-q_0)}{r_0^3} - \frac{(\cancel{1+}\mu)(q-q_1)}{r_1^3}$$

where $r_0^2 = (X - X_0)^2 + (Y - Y_0)^2 + (Z - Z_0)^2$

$$r_1^2 = (X - X_1)^2 + (Y - Y_1)^2 + (Z - Z_1)^2$$

②



Two bodies m_0 and m_1 are moving uniformly with constant angular speed. Thus it is natural to consider the rotating frame where the primaries locations are fixed.

Let (x, y, z) be the coords of the ~~particle~~ massless body m in the rotating (synodic) system.

Locations of P_0 : $(x_0, y_0, z_0) = (-\mu, 0, 0)$

P_1 : $(x_1, y_1, z_1) = (1-\mu, 0, 0)$

The relation between (X, Y, Z) and (x, y, z) is

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

From which one has

$$\begin{pmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{pmatrix} = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \dot{x} - y \\ \dot{y} + x \\ \dot{z} \end{pmatrix}$$

and

$$\begin{pmatrix} \ddot{X} \\ \ddot{Y} \\ \ddot{Z} \end{pmatrix} = \begin{pmatrix} \cos t & -\sin t & \\ \sin t & \cos t & \\ & & 1 \end{pmatrix} \begin{pmatrix} \ddot{x} - 2\dot{y} - x \\ \ddot{y} + 2\dot{x} - y \\ \ddot{z} \end{pmatrix}$$

substituting these into equation of motion in inertial frame one has (after some calculation and simplification)

$$\ddot{x} - 2\dot{y} - x = -\left[\frac{(1-\mu)(x+\mu)}{r_0^3} + \mu \frac{x-1+\mu}{r_1^3}\right]$$

$$\ddot{y} + 2\dot{x} - y = -\left[\frac{1-\mu}{r_0^3} + \frac{\mu}{r_1^3}\right] y$$

$$\ddot{z} = -\left[\frac{1-\mu}{r_0^3} + \frac{\mu}{r_1^3}\right] z$$

define the effective potential

$$U = -\frac{1}{2}(x^2+y^2) \cancel{+} - \frac{1-\mu}{r_0} - \frac{\mu}{r_1}$$

where the last two terms are gravitational potential.

then the eqns of motion are

$$\ddot{x} - 2\dot{y} = -\frac{\partial U}{\partial x}$$

$$\ddot{y} + 2\dot{x} = -\frac{\partial U}{\partial y}$$

$$\ddot{z} = -\frac{\partial U}{\partial z}$$

Here we consider spatial case, by letting $z = \dot{z} = 0$ we have the planar case.

Jacobi constant:

$$C_J = -(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \cancel{\mp} 2U$$

$$= -(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \cancel{2}(x^2 + y^2) + 2\left(\frac{1-\mu}{r_0} + \frac{\mu}{r_1}\right)$$

The system is not integrable since this is the only integral.

But this integral gives us some information about the problem.

Indeed

$$C_J = -v^2 - 2U$$

we have

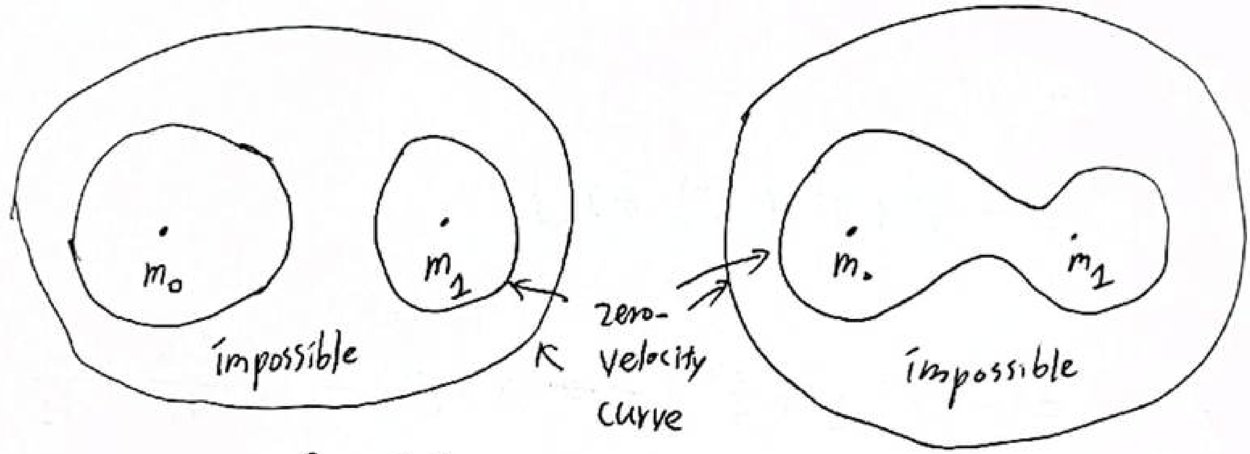
$$v^2 = -2U - C_J \geq 0$$

that is

$$x^2+y^2+2\frac{1-\mu}{r_0}+2\frac{\mu}{r_1} \geq C_J ,$$

note this inequality defines the admissible region in the configuration space.

The boundary of this region is called zero-velocity surface.
(or zero-velocity curve 對 for planar case.)



$\mu = 0.2$, $C_J = 3.9$

motion is restricted to the region around m_0 or m_1

$\mu = 0.2$, $C_J = 3.7$

motion is restricted to region including m_0, m_1

- There are 5 equilibrium pts where the slope is zero;

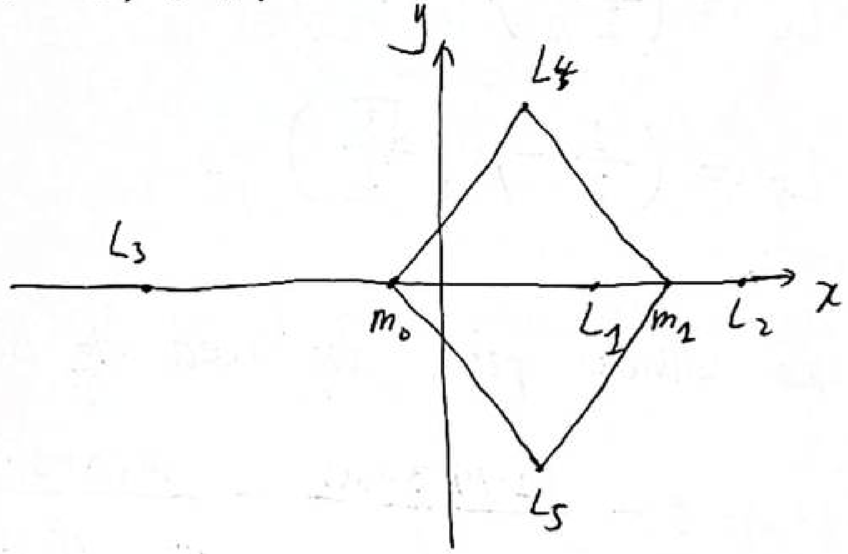
since $-2U$ is symmetric w.r.t. y-axis, we have several equilibrium pts located at x-axis, and several equil. pts. distributed symmetrically with x-axis.

(4) These 5 ~~equilibrium~~ critical pts correspond to equilibrium pts of the problem, i.e., if equil. pts are (x_i, y_i) $i=1,\cdots,5$, then the equil. pts of the eqns of motion are

$$(x_i, y_i, 0, 0) \quad , i=1,\cdots,5$$

Indeed: there are

- 3 collinear ~~crit~~ equilibria on x-axis, called L_1, L_2, L_3
- 2 equilateral pts L_4, L_5.



First, we consider ~~[illegible]~~ the equilibria for $y \neq 0$.

$$\frac{\partial U}{\partial x} = -x + \frac{(1-\mu)(x+\mu)}{r_0^3} + \frac{\mu(x-1+\mu)}{r_1^3} = 0$$

$$\frac{\partial U}{\partial y} = -y + \frac{(1-\mu)y}{r_0^3} + \frac{\mu y}{r_1^3} = 0$$

$$= -1 + \frac{(1-\mu)}{r_0^3} + \frac{\mu}{r_1^3} = 0$$

From which one can obtain

$$\begin{cases} \dfrac{1}{r_0^3} + \dfrac{1}{r_1^3} = 1 \\[2ex] \dfrac{1}{r_0^3} - \dfrac{1}{r_1^3} = 0 \end{cases}$$

$$\Rightarrow \quad r_0 = r_1 = 1$$

and this gives exactly the equilateral equilibria.

$$L_4 := \left(\frac{1}{2} - \mu, \frac{\sqrt{3}}{2} \right)$$

$$L_5 := \left(\frac{1}{2} - \mu, \frac{\sqrt{3}}{2} \right)$$

For the collinear pts, we need to solve

$$- \frac{\partial u}{\partial x}(x,0) = x - \frac{(1-\mu)(x+\mu)}{r_0^3} - \frac{\mu(x-1+\mu)}{r_1^3} = 0$$

$$= x - \frac{(1-\mu)(x+\mu)}{|x+\mu|^3} - \frac{\mu(x-1+\mu)}{|x-1+\mu|^3} = 0$$

We will have a ~~qua~~ quintic equation which is difficult to solve. But we can quanlitatively analyse it.

5

let $y=0$, then

$$U(x,0) = -\frac{1}{2}x^2 - \frac{1-\mu}{|x+\mu|} - \frac{\mu}{|x-1+\mu|}$$

one can notice that

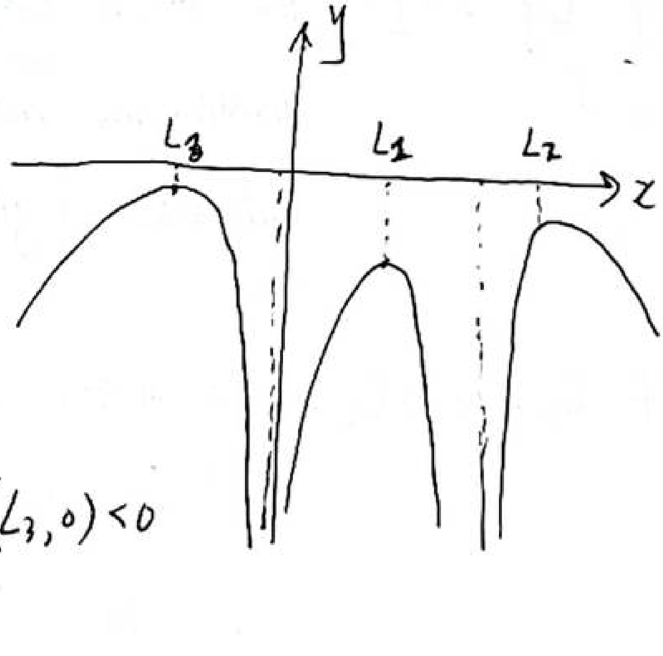
as x goes to $-\infty$, $-\mu$, $1-\mu$, or $+\infty$, value of $U(x,0)$ goes to $-\infty$.

Besides:

$$U''(x,0) = -1 - \frac{2(1-\mu)}{|x+\mu|^3} - \frac{2\mu}{|x-1+\mu|^3} < 0$$

hence $U(x,0)$ is concave in each interval.

Therefore, $U(x,0)$ has 3 critical pts, ~~from left to right~~, denoted as L_1, L_2, L_3



If $\mu \in (0, \frac{1}{2})$:

$$U(L_1,0) < U(L_2,0) < U(L_3,0) < 0$$

If $\mu \in (\frac{1}{2}, 0)$

$$U(L_1,0) < U(L_3,0) < U(L_2,0) < 0$$

Let's denote C_1, C_2, C_3, C_4, C_5 the Jacobi constant of the asteroid "rest" at the equilibrium pts.

$$C_1 > C_2 > C_3 > C_4 = C_5$$

If we define the following set

$$\mathcal{H}(C_J) = \{(x,y) : -2U(x,y) \geq C_J\},$$

called Hill's region for C_J, ~~wh~~ and the boundary of this region is the zero-vel. curves.

For a specific orbit C_J (which is constant once the initial condition is given):

if $C_J > C_1$: the motion can be in two bounded regions around m_0 and m_1, respectively, or can be in the unbounded region. $\mathcal{H}(C_J) = \mathcal{H}^{m_0} \cup \mathcal{H}^{m_1} \cup \mathcal{H}^{u}$

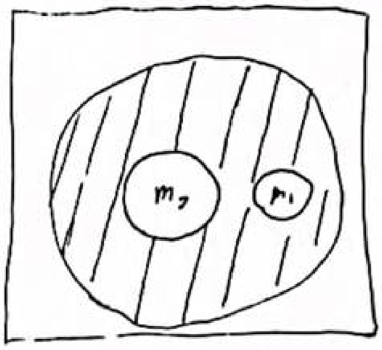
if $C_1 > C_J > C_2$: the motion can be inside the bounde region which contains both m_0 and m_1, or can be in the unbounded region.

$$\mathcal{H}(C_J) = \mathcal{H}^{m_0, m_1} \cup \mathcal{H}^{u}$$

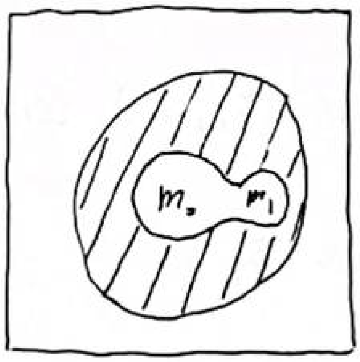
⑥ if $C_2 > C_J > C_3$: the motion will only be in the unbounded region. The forbidden region is connected.

if $C_3 > C_J > C_4$: the motion is in the unbounded region, and the forbidden regions are around L_4, L_5

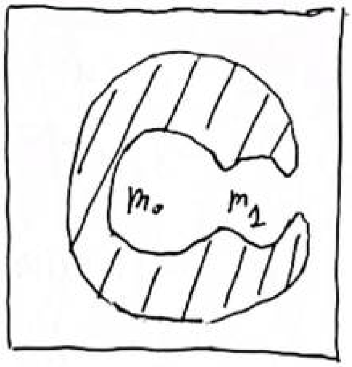
if $C_4 > C_J$: the entire space is hill's region.



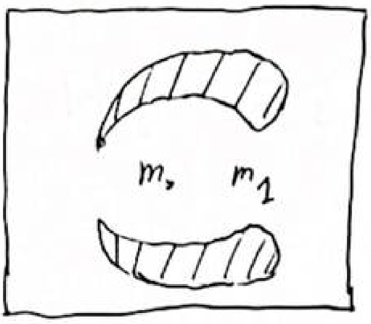
$C_J > C_1$



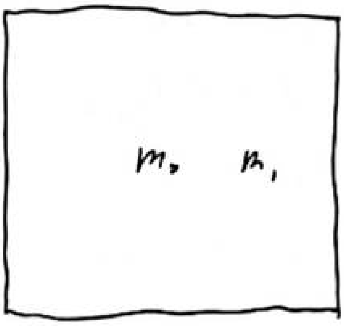
$C_1 > C_J > C_2$



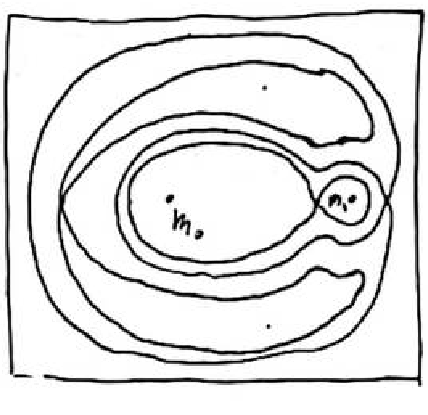
$C_2 > C_J > C_3$



$C_3 > C_J > C_4$



$C_4 > C_J$



Dynamics of the linearized system at the Lagrangian pts

recall that eqns of motion are

$$\ddot{x} - 2\dot{y} = -\frac{\partial U}{\partial x}(x, y)$$

$$\ddot{y} + 2\dot{x} = -\frac{\partial U}{\partial y}(x, y)$$

we consider the following situation: we consider words

$x = x_i + u$
$y = y_i + v$, where (x_i, y_i) correspond to Lag. pts,

after substitution, we expand the r.h.s. around (x_i, y_i)

and ignore higher order terms.

$$\ddot{x}_i + \ddot{u} - 2\dot{y}_i - 2\dot{v} = -\frac{\partial U}{\partial x}(x_i, y_i) - \frac{\partial^2 U}{\partial x^2}(x_i, y_i)\, u - \frac{\partial^2 U}{\partial x \partial y}(x_i, y_i)\, v$$

$$\ddot{y}_i + \ddot{v} + 2\dot{x}_i + 2\dot{u} = -\frac{\partial U}{\partial y}(x_i, y_i) - \frac{\partial^2 U}{\partial y \partial x}(x_i, y_i)\, u - \frac{\partial^2 U}{\partial y^2}(x_i, y_i)\, v$$

due to the definition of Lag. pts, we obtain

$$\ddot{u} - 2\dot{v} = -\frac{\partial^2 U}{\partial x^2}(x_i, y_i)\, u - \frac{\partial^2 U}{\partial x \partial y}(x_i, y_i)\, v$$

$$\ddot{v} + 2\dot{u} = -\frac{\partial^2 U}{\partial y \partial x}(x_i, y_i)\, u - \frac{\partial^2 U}{\partial y^2}(x_i, y_i)\, v$$

which is the linearized 2-nd order DE.

3 ~~denote: $v_u = \dot{u}$,~~

~~$Vel_u = \dot{u}$, $Vel_v = \dot{v}$~~

~~then the 1-order eqns are:~~

~~$\dot{u} = \dot{Vel_u}$~~

~~$\ddot{v} =$~~

denote $\xi = \dot{u}$, $\eta = \dot{v}$, then ther 1st order ODEs are

$$\dot{u} = \xi$$

$$\dot{v} = \eta$$

$$\dot{\xi} = 2\eta - \frac{\partial^2 U}{\partial x^2} u - \frac{\partial^2 U}{\partial x \partial y} v$$

$$\dot{\eta} = -2\xi - \frac{\partial^2 U}{\partial y \partial x} u - \frac{\partial^2 U}{\partial y^2} v$$

$$\Leftrightarrow \begin{pmatrix} \dot{u} \\ \dot{v} \\ \dot{\xi} \\ \dot{\eta} \end{pmatrix} = A \begin{pmatrix} u \\ v \\ \xi \\ \eta \end{pmatrix}$$

Where A , called the Jacobian matrix, is

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -U_{xx} & -U_{xy} & 0 & 2 \\ -U_{xy} & -U_{yy} & -2 & 0 \end{pmatrix}$$

note the matrix is evaluated at the Lag. pts.

To understand the dynamics we need to compute the eigenvalues and eigenvectors.

the eigenvalues can be computed by

$$\det(\lambda I - A) = 0$$

$$\Leftrightarrow \det\left(\begin{pmatrix} \lambda & \\ & \lambda \end{pmatrix} \begin{pmatrix} \lambda & -2 \\ 2 & \lambda \end{pmatrix} - \begin{pmatrix} U_{xx} & U_{xy} \\ U_{xy} & U_{yy} \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right) = 0$$

$$\Leftrightarrow \det\left(\begin{pmatrix} \lambda^2 & -2\lambda \\ 2\lambda & \lambda^2 \end{pmatrix} + \begin{pmatrix} U_{xx} & U_{xy} \\ U_{xy} & U_{yy} \end{pmatrix} \right) = 0$$

Note:

$$U_{xx} = -1 + \frac{1-\mu}{r_0^3} + \frac{\mu}{r_1^3} - 3\frac{(1-\mu)(x+\mu)^2}{r_0^5} - 3\frac{\mu}{r_1^5}(x-1+\mu)^2$$

$$U_{xy} = -3\frac{(1-\mu)(x+\mu)y}{r_0^5} - 3\frac{\mu(x-1+\mu)y}{r_1^5}$$

$$U_{yy} = -1 + \frac{1-\mu}{r_0^3} + \frac{\mu}{r_1^3} - 3\frac{(1-\mu)y^2}{r_0^5} - 3\frac{\mu y^2}{r_1^5}$$

First we consider L_4, L_5, it is enough to consider only L_4.

$$L_4 = \left(\frac{1}{2} - \mu, \frac{\sqrt{3}}{2}\right)$$

and
$$\partial^2 U = \begin{pmatrix} -\frac{3}{4} & +\frac{3\sqrt{3}}{4}(-1+2\mu) \\ \frac{3\sqrt{3}}{4}(-1+2\mu) & -\frac{9}{4} \end{pmatrix}$$

④ then the eigenvalues can be obtained from:

$$\det\begin{pmatrix} \lambda^2-\frac{3}{4} & \frac{3\sqrt{3}}{4}(-1+2\mu)-2\lambda \\ \frac{3\sqrt{3}}{4}(-1+2\mu)+2\lambda & \lambda^2-\frac{9}{4} \end{pmatrix}=0$$

$$\Leftrightarrow \left(\lambda^2-\frac{3}{4}\right)^{2}-\left(\frac{3\sqrt{3}}{4}(-1+2\mu)-2\lambda\right)\left(\frac{3\sqrt{3}}{4}(-1+2\mu)+2\lambda\right)=0$$

$$\left(\lambda^2-\frac{9}{4}\right)$$

$$\Leftrightarrow \quad \lambda^4+\lambda^2+\frac{27}{4}\mu-\frac{27}{4}\mu^2=0$$

$$\Leftrightarrow \quad \lambda^2=\frac{-1\pm\sqrt{1-27\mu+27\mu^2}}{2}$$

define $\mu_R=\dfrac{9-\sqrt{69}}{18}\sim 0.038$ called Routh's critical value

then for $0<\mu<\mu_R$, we have $27\mu^2\mp 27\mu+1>0$.

And for $0<\mu<\mu_R$, we have

$$\frac{-1\pm\sqrt{1-27\mu+27\mu^2}}{2}<0 \text{ are real numbers}$$

then we have 4 complex eigenvalues $\lambda=\pm i\omega_1$, $\lambda=\pm i\omega_2$

where

$$\omega_1=\sqrt{\frac{+1+\sqrt{1-27\mu+27\mu^2}}{2}}, \quad \omega_2=\sqrt{\frac{+1-\sqrt{1-27\mu+27\mu^2}}{2}}$$

therefore, equilateral equilibria are stable for $0<\mu<\mu_R$.

Now, let's consider collinear pts L_1, L_2, L_3,

at collinear pts, we have that

$$y=0, \quad r_0 = |x_i+\mu|, \quad r_1 = |x_i-1+\mu|$$

then

$$U_{xy}=0$$

$$U_{xx} = -1-2\frac{1-\mu}{r_0^3}-2\frac{\mu}{r_1^3} =: \alpha_1 < 0$$

[we will not prove this.]

$$U_{yy} = -1+\frac{1-\mu}{r_0^3}+\frac{\mu}{r_1^3} =: \alpha_2 > 0$$

Correspondingly,

$$\partial^2 U = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}$$

The eigenvalues can be computed by

$$\det\begin{pmatrix} \lambda^2+\alpha_1 & -2\lambda \\ 2\lambda & \lambda^2+\alpha_2 \end{pmatrix} = 0$$

$$\Rightarrow \quad \lambda^4+\lambda^2(\alpha_1+\alpha_2+4)+\alpha_1\alpha_2 = 0$$

$$\Rightarrow \quad \lambda^2 = \frac{-(\alpha_1+\alpha_2+4)\pm\sqrt{(\alpha_1+\alpha_2+4)^2-4\alpha_1\alpha_2}}{2}$$

⑨ clearly, we have

$$(\alpha_1^2+\alpha_2^2+4)^2-4\alpha_1\alpha_2>0$$

then we have real eigenvalues

$$\lambda_{1,2}=\pm\sqrt{\frac{-(\alpha_1+\alpha_2+4)+\sqrt{(\alpha_1+\alpha_2+4)^2-4\alpha_1\alpha_2}}{2}}$$

and purely imaginary eigenvalues

$$\lambda_{3,4}=\pm i\sqrt{\frac{+(\alpha_1+\alpha_2+4)+\sqrt{(\alpha_1+\alpha_2+4)^2-4\alpha_1\alpha_2}}{2}}$$

<u>Prop:</u> If all ~~ei~~ the ~~egi~~ eigenvalues of A have a strictly negative real part, then the equilibrium is stable.

If A has ~~one~~ at least one eigenvalue with strictly positive real part, then the equilibrium is unstable.

<u>Def:</u> The equilibrium pt is said to be:

(i) elliptic, if all the ~~real~~ eigenvalues of A have zero real part;

(ii) hyperbolic, if all the eigenvalues of A have non-zero real parts;

(iii) partially hyperbolic, otherwise.

For different eigenvalues λ, we have corresponding eigenvector

u_λ

We denote: E^s stable space, spanned by u_λ if $Re(\lambda)<0$;

E^c center space, spanned by u_λ if $Re(\lambda)=0$;

E^u unstable space, spanned by u_λ if $Re(\lambda)>0$.

Remark: the entire space can be regarded as sum of

E^s, E^c and E^u.

Dynamics of the linearized system

Now the linearized system is

$$\begin{pmatrix} \dot{u} \\ \dot{v} \\ \dot{\xi} \\ \dot{\eta} \end{pmatrix} = A \begin{pmatrix} u \\ v \\ \xi \\ \eta \end{pmatrix}$$

Where A is evaluated at the Lag. pts L_i.

For equilateral equilibria L_4 and L_5, we have its eigenvalues

$\pm i\omega_1$, $\pm i\omega_2$, correspondingly, we have eigenvector

$\Omega_1, \bar{\Omega}_1, \Omega_2, \bar{\Omega}_2$ $\qquad P = (\Omega_1, \bar{\Omega}_1\ \Omega_2\ \bar{\Omega}_2)$

introduce $\begin{pmatrix} u \\ v \\ \xi \\ \eta \end{pmatrix} = P \begin{pmatrix} \hat{u} \\ \hat{v} \\ \hat{\xi} \\ \hat{\eta} \end{pmatrix}$ then $\begin{pmatrix} \dot{\hat{u}} \\ \dot{\hat{v}} \\ \dot{\hat{\xi}} \\ \dot{\hat{\eta}} \end{pmatrix} = P^{-1}AP \begin{pmatrix} \hat{u} \\ \hat{v} \\ \hat{\xi} \\ \hat{\eta} \end{pmatrix}$

(20)

and $P^{-1}AP = \begin{pmatrix} i\omega_1 & & & \\ & -i\omega_1 & & \\ & & i\omega_2 & \\ & & & -i\omega_2 \end{pmatrix}$

then $\hat{u} = \hat{u}(0)\, e^{i\omega_1 t}$ where $\hat{u}(0) = \overline{\hat{v}(0)}$

$\hat{v} = \hat{v}(0)\, e^{-i\omega_1 t}$

$\hat{\xi} = \hat{\xi}(0)\, e^{i\omega_2 t}$ $\quad \hat{\xi}(0) = \overline{\hat{\eta}(0)}$

$\hat{\eta} = \hat{\eta}(0)\, e^{-i\omega_2 t}$

Because $u = \Omega_1 \hat{u} + \bar{\Omega}_1 \hat{v} + \Omega_2 \hat{\xi} + \bar{\Omega}_2 \hat{\eta}$

real

$\bar{u} = \bar{\Omega}_1 \bar{\hat{u}} + \Omega_1 \bar{\hat{v}} + \bar{\Omega}_2 \bar{\hat{\xi}} + \Omega_2 \bar{\hat{\eta}}$

$\bar{u} = u \Rightarrow \hat{u} = \bar{\hat{v}}$, $\quad \hat{\xi} = \bar{\hat{\eta}}$

$\{\Omega_1, \bar{\Omega}_1, \Omega_2, \bar{\Omega}_2\}$ is the basis (independent)

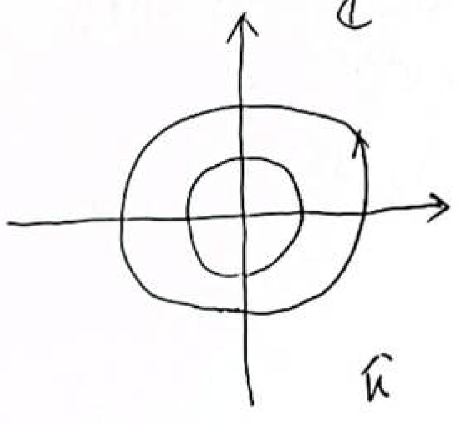
$\overline{\hat{u}} = \hat{v}$

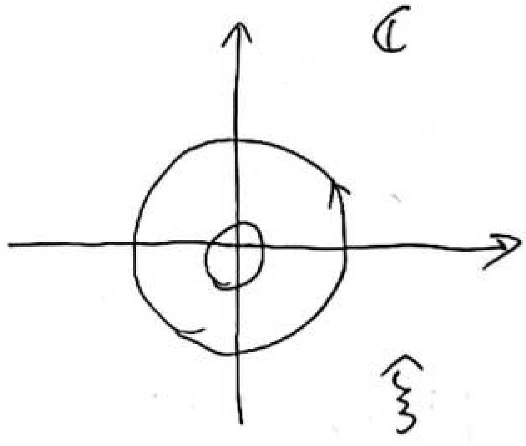
$\overline{\hat{\xi}} = \hat{\eta}$

$\Downarrow$

$\hat{u}\,\bar{\hat{u}} = |\hat{u}(0)|^2$ and $\hat{\xi}\,\bar{\hat{\xi}} = |\hat{\xi}(0)|^2$

in complex plane





different radius correspond to different initial conditions therefore, for the linearized system, the ~~subspace~~ space there is center space

$$E^{c} = \underset{\substack{\nearrow \\ \text{related to} \\ \pm\omega_1}}{E^{c,1}} \oplus \underset{\substack{\nwarrow \\ \text{related to} \\ \pm\omega_2}}{E^{c,2}}$$

We say L_4, L_5 is center $\times$ center (for $\mu < \mu_R$)

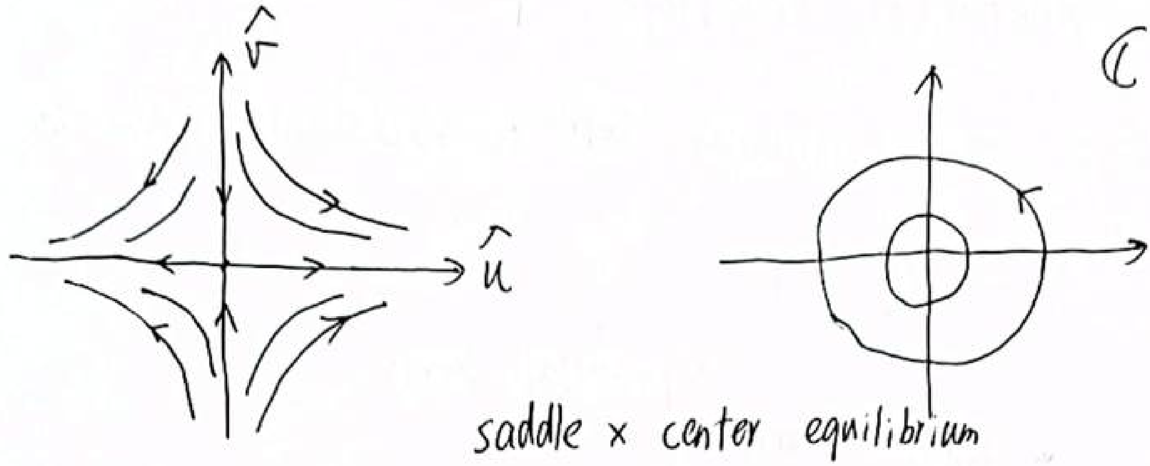
Now for collinear equilibria L_1, L_2, and L_3, we have real and purely imaginary eigenvalues $\lambda_1 > 0$, $\lambda_2 < 0$, λ_3, λ_4

we can similarly introduce the new coords by

$$\begin{pmatrix} u \\ v \\ \xi \\ \eta \end{pmatrix} = P \begin{pmatrix} \hat{u} \\ \hat{v} \\ \hat{\xi} \\ \hat{\eta} \end{pmatrix}$$

then we have (note $\lambda_1 = -\lambda_2$, $\lambda_3 = \bar{\lambda}_4$)

$$\begin{aligned} \dot{\hat{u}} &= \lambda_1 \hat{u} \\ \dot{\hat{v}} &= \lambda_2 \hat{v} \\ \dot{\hat{\xi}} &= \lambda_3 \hat{\xi} \\ \dot{\hat{\eta}} &= \lambda_4 \hat{\eta} \end{aligned} \Rightarrow \begin{aligned} &\left.\begin{aligned} \hat{u} &= \hat{u}(0) e^{\lambda_1 t} \\ \hat{v} &= \hat{v}(0) e^{-\lambda_1 t} \end{aligned}\right\} \text{saddle} \\ &\left.\begin{aligned} \hat{\xi} &= \hat{\xi}(0) e^{\lambda_3 t} \\ \hat{\eta} &= \hat{\eta}(0) e^{\bar{\lambda}_3 t} \end{aligned}\right\} \text{center} \end{aligned}$$



saddle × center equilibrium

We have the following situations.

(i) if both $(\hat{u},\hat{v})=(0,0)$ and $(\hat{\xi},\hat{\eta})=(0,0)$, then we have the equilibrium solution;

(ii) fixing $(\hat{u},\hat{v})=(0,0)$, but $(\hat{\xi},\hat{\eta})\neq(0,0)$, then we obtain a family of periodic orbits; $\to$ denoted as Θ

(iii) consider $\hat{u}=0$, $\hat{v}\neq 0$, $(\hat{\xi},\hat{\eta})\neq(0,0)$, the orbit will converge to Θ in the future;

(iv) consider $\hat{u}\neq 0$, $\hat{v}=0$, $(\hat{\xi},\hat{\eta})\neq(0,0)$, the orbit will converge to Θ in the past;

(v) if $(\hat{u},\hat{v})\neq(0,0)$, we say the orbit is satterred away from the origin of $(\hat{u},\hat{v})$ plane.

Hamiltonian function of PCRTBP

Recall that the Lagrangian of a mechanical system is

$$L = \underbrace{K}_{\text{kinetic energy}} - \underbrace{P}_{\text{potential energy}}$$

then the eqns of motion can be obtained from Euler-Lag equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0$$

If introduce conjugate momentum

$$p = \frac{\partial L}{\partial \dot{q}}$$

E-L eqns are equivalent to Hamilton's eqns

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q}$$

with $H(q, p, t) = p^T \dot{q} - L(q, \cancel{R}, \dot{q}, t)$ by replacing $\dot{q}$ with $\dot{q}(q, p)$.

(12) Now recall that:

In the inertial coords frame, kinetic energy

$$K = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

and the potential ~~a~~ energy is

$$\mathcal{U}(x, y, z, t) = -\frac{1-\mu}{r_0} - \frac{\mu}{r_1}$$

~~Now we consider the rotating frame, then~~

Hence the Lagrangian is, in the inertial frame,

$$L(x, y, z, \dot{x}, \dot{y}, \dot{z}, t) = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{1-\mu}{r_0} + \frac{\mu}{r_1}$$

In the rotating frame:

$$L(x, y, z, \dot{x}, \dot{y}, \dot{z}) = \frac{1}{2}\left((\dot{x}-y)^2 + (\dot{y}+x)^2 + \dot{z}^2\right) - \overset{\mathcal{U}(x, y, z)}{\cancel{\mathcal{U}(x, y, z)}}$$

Introduce the conjugate momentum:

$$p_x = \frac{\partial L}{\partial \dot{x}} = \dot{x} - y \qquad \Rightarrow \quad \dot{x} = p_x + y$$

$$p_y = \frac{\partial L}{\partial \dot{y}} = \dot{y} + x \qquad \Rightarrow \quad \dot{y} = p_y - x$$

$$p_z = \frac{\partial L}{\partial \dot{z}} = \dot{z} \qquad \Rightarrow \quad \dot{z} = p_z$$

Then the Hamiltonian function is

$$H(x,y,z,p_x,p_y,p_z) = p_x\dot{x} + p_y\dot{y} + p_z\dot{z} - L$$

$$= p_x(p_x + y) + p_y(p_y - x) + p_z^2 - \frac{1}{2}(p_x^2 + p_y^2 + p_z^2) +$$

$$+ \mathcal{U}(x,y,z)$$

$$= \frac{1}{2}(p_x^2 + p_y^2 + p_z^2) + yp_x - xp_y + \mathcal{U}(x,y,z)$$

$$= \frac{1}{2}\left((p_x+y)^2 + (p_y - x)^2 + p_z^2\right) + U(x,y,z)$$

recall that the effective potential is

$$U(x,y,z) = -\frac{1}{2}(x^2+y^2) - \frac{1-\mu}{r_0} - \frac{\mu}{r_1}$$

Levi-Civita Regularization. (For PCRTBP)

Why? We want to study dynamics where the orbit of the massless body collides with the primaries.

(we consider the smaller one in the notes)

the potential is singular : $-\dfrac{1-\mu}{r_0} - \dfrac{\mu}{r_2}$

eqns of motion

$$\Big\downarrow \text{regularization}$$

the eqns of motion are regular at "singular pts".

(13) three key ingredients:

(i) Fictitious time

(ii) Conformal squaring

(iii) Fixed energy hypersurface

We first explore the conformal squaring:

regard $\mathbb{R}^2$ as $\mathbb{C}$: $(x, y) \cong \overset{=: q}{x+iy} \in \mathbb{C}$

call this complex physical plane

the similarly, $(p_x, p_y) \cong \underset{=: p}{p_x + i p_y} \in T^*_{\underset{(x+iy)}{q}}\mathbb{C} \cong \mathbb{C}$

and the symplectic 2-form is

$$dp_x \wedge dx + dp_y \wedge dy = \mathrm{Re}\left(d(p_x + i p_y) \wedge d(x - iy)\right)$$

the Levi-Civita map is define as

$$\text{L.C.} : T^*(\mathbb{C}\setminus\{0\}) \to T^*(\mathbb{C}\setminus\{0\})$$

$$(z, w) \mapsto \left(z^2, \frac{w}{2\bar{z}}\right) = (x+iy, p_x + i p_y)$$

$$= (q, p)$$

One can verify that, L.C. is symplectic, i.e.

$$\cancel{dq \wedge d} \ \mathrm{Re}(dp \wedge d\bar{q}) = \mathrm{Re}\left(\frac{2\bar{z}\,dw - 2w\,d\bar{z}}{4\bar{z}^2} \wedge 2\bar{z}\,d\bar{z}\right) = \mathrm{Re}(dw \wedge d\bar{z})$$

Forgot Step : we consider the following change of coords

$$x := \tilde{x} + (1-\mu) \qquad p_x := \tilde{p}_x$$

canonical

$$y := \tilde{y} \qquad p_y := \tilde{p}_y + (1-\mu)$$

When $x = 1-\mu \Rightarrow \tilde{x} = 0$, we shift the ~~se~~ smaller primary to the origin.

The new Hamiltonian is, in planar case,

$$\tilde{H} = \frac{1}{2}\left(\tilde{p}_x^2 + (\tilde{p}_y + (1-\mu))^2\right) + \tilde{y}\tilde{p}_x - (\tilde{x} + (1-\mu))(\tilde{p}_y + (1-\mu))$$

$$- \frac{1-\mu}{\sqrt{(\tilde{x}+1)^2 + \tilde{y}^2}} - \frac{\mu}{\sqrt{\tilde{x}^2 + \tilde{y}^2}}$$

$$= \frac{1}{2}\left(\tilde{p}_x^2 + \tilde{p}_y^2\right) + \tilde{y}\tilde{p}_x - \tilde{x}\tilde{p}_y - \frac{1-\mu}{\sqrt{(\tilde{x}+1)^2 + y^2}} - \frac{\mu}{\sqrt{\tilde{x}^2 + \tilde{y}^2}} - (1-\mu)\tilde{x}$$

$$- \frac{(1-\mu)^2}{2}$$

From now on, we will drop "~" for simplicity.

Now, back to Levi-Civita regularization

denote $z = z_1 + i z_2$, $\quad w = w_1 + i w_2$

then $z^2 = q = x + iy$, $\quad \frac{w}{2\bar{z}} = p_x + i p_y$

we get

$$\begin{cases} x = z_1^2 - z_2^2 \\ y = 2 z_1 z_2 \end{cases} \qquad \begin{cases} p_x = \dfrac{z_1 w_1 - z_2 w_2}{2(z_1^2 + z_2^2)} \\ p_y = \dfrac{z_2 w_1 + z_1 w_2}{2(z_1^2 + z_2^2)} \end{cases}$$

(74) Now we need to substitute them and do the long computations

$$H = \frac{1}{8}\left(\frac{(z_1 w_1 - z_2 w_2)^2}{|z|^4} + \frac{(z_2 w_1 + z_1 w_2)^2}{|z|^4} \right) + 2 z_1 z_2 \frac{z_1 w_2 - z_2 w_2}{2|z|^2}$$

$$- (z_1^2 - z_2^2) \frac{z_2 w_1 + z_1 w_2}{2|z|^2} - \frac{1-\mu}{\sqrt{|z|^4 + 2(z_1^2 - z_2^2) + 1}} - \frac{\mu}{|z|^2}$$

$$- (1-\mu)(z_1^2 - z_2^2) - \frac{(1-\mu)^2}{2}$$

$$\cancel{= \frac{1}{8|z|^2}} \left(\begin{array}{l} 2 z_1^2 z_2 w_1 - 2 z_1 z_2^2 w_2 - z_1^2 z_2 w_1 - z_1^3 w_2 + z_2^3 w_1 + z_2^2 z_1 w_2 \\ z_1^2 z_2 w_1 - z_1^3 w_2 + z_2^3 w_1 - z_1 z_2^2 w_2 \end{array} \right.$$

$$= \frac{|w|^2}{8|z|^2} + \frac{1}{2}(z_2 w_1 - z_1 w_2) - \frac{\mu}{|z|^2} - (1-\mu)\left(\frac{1}{\sqrt{|z|^4 + 2(z_1^2 - z_2^2) + 1}} + z_1^2 - z_2^2 \right)$$

where we ~~drop~~ ignore the constant term $-\frac{(1-\mu)^2}{2}$.

— o —

Before moving forward, we consider a different way to get the canonical transformation induced by complex squaring.

The map $z \mapsto z^2$ gives us the transformation of "position" coords, we now have the coords z_1, z_2, x, y, p_x, p_y, we have to identify the conjugate momentum of z_1, z_2 (i.e. w_1, w_2)

We consider the generating function

$$S_2 = (z_1^2 - z_2^2) p_x + 2 z_1 z_2 p_y = \cancel{S_2(z, p)}\ S_2(p, z)$$

from which, we have

$$\frac{\partial^2 S_2}{\partial p \partial z} = \begin{pmatrix} 2z_1 & -2z_2 \\ 2z_2 & 2z_1 \end{pmatrix}$$

and ~~ato~~ according to the def of generating function

$$x = \frac{\partial S_2}{\partial p_x} = z_1^2 - z_2^2$$

what we've already known

$$y = \frac{\partial S_2}{\partial p_y} = 2z_1 z_2$$

and

$$w_1 = \frac{\partial S_2}{\partial z_1} = 2z_1 p_x + 2z_2 p_y$$

$$w_2 = \frac{\partial S_2}{\partial z_2} = -2z_2 p_x + 2z_1 p_y$$

inverting it we have

$$p_x = \frac{z_1 w_1 - z_2 w_2}{2(z_1^2 + z_2^2)}$$

what we need to get

$$p_y = \frac{z_2 w_1 + z_1 w_2}{2(z_1^2 + z_2^2)}$$

—o—

Now we move back to the regularization,

we consider a specific energy surface $\{H = E\}$

over which collision may happen, i.e., $|z|^2 = 0$

To eliminate the singularity, we introduce fictitious time τ

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and time reparameterization:

$$dt = |z|^2 d\tau \quad , \text{ equivalently }, \quad \frac{d}{d\tau} = |z|^2 \frac{d}{dt}$$

The regularized Hamiltonian will be:

$$K_E = |z|^2 (H - E)$$

$$= \frac{1}{8}|w|^2 + \frac{|z|^2}{2}(z_2 w_1 - z_1 w_2) - |z|^2\left(\frac{(1-\mu)^2}{2} + E\right) - \mu$$
$$- (1-\mu)|z|^2\left(\frac{1}{\sqrt{|z|^4 + 2(z_1^2 - z_2^2) + 1}} + z_1^2 - z_2^2\right)$$

Now, there is no collision singularity w.r.t. smaller primary, but collision singularity w.r.t. larger primary still exists.

The time reparameterization changes the "velocity" of the vector field.

Indeed,

$$X_{K_E} = J \nabla K_E$$

$$= J \nabla\left(|z|^2 (H - E)\right)$$

$$= J(\nabla |z|^2)(H - E) + J|z|^2 \nabla(H - E)$$

evaluated at $\{H=E\}$ $$\stackrel{!}{=} J|z|^2 \nabla(H - E) = |z|^2 J \nabla H = |z|^2 X_H$$

So on the energy hypersurface

$$\{K_E = 0\} \equiv \{H = E\}$$

We have

$$X_{K_E} = |z|^2 X_H$$

close to the singularity, $|z|^2 \to 0$, hence $|z|^2$ plays a role of slowing down the vector field.

Moreover, the v.f. in the left-hand side has no singularity $\Rightarrow$ the flow is "extended" to the singularity.

Remark:

- Levi-Civita map is 2-to-1 covering mapping.
- One can also use Levi-Civita regularization to kepler problem to regularized the collision with the origin.
- L.C. regularization only works for planar case, for spatial case one needs to use Kustaanheimo-Stiefel regularization
- For kepler problem, after regularization, the problem is transformed into a Harmonic oscillator.

(26) Now, Let's derive the equations of motion of regularized System:

$$z_1'' = \frac{1}{4}\left[(a+b)\,z_1 + c\,z_2\right]$$

$$z_2'' = \frac{1}{4}\left[(a-b)\,z_1 + c\,z_2\right]$$

where:

$$a = \frac{2(1-\mu)}{r_0} - E + x^2 + y^2$$

$$b = 4y' + 2r_1 x - \frac{2(1-\mu)\,r_1(x+\mu)}{r_0^3}$$

$$c = 2r_1 y - 4x' - \frac{2(1-\mu)\,r_1 y}{r_0^3}$$